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Modules which are coinvariant under automorphisms of their projective covers

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In this paper we study modules coinvariant under automorphisms of their projective covers. We first provide an alternative, and in fact, a more succinct and conceptual proof for the result that a module M is invariant under automorphisms of its injective envelope if and only if given any submodule N of M, any monomorphism $f : N \to M$ can be extended to an endomorphism of M and then, as a dual of it, we show that over a right perfect ring, a module M is coinvariant under automorphisms of its projective cover if and only if for every submodule N of M, any epimorphism $\varphi : M \to M/N$ can be lifted to an endomorphism of M.

1. Introduction

Modules invariant or coinvariant under automorphisms of their covers or envelopes have been recently introduced in [7]. Recall that a class $\mathcal{X}$ of right modules over a ring R,

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closed under isomorphisms, is called an *enveloping class* if for any right R-module M, there exists a homomorphism $u : M \to X(M)$, with $X(M) \in \mathcal{X}$, such that any other morphism from M to a module in $\mathcal{X}$ factors through u and, moreover, whenever u has a factorization $u = h \circ u$, then h must be an automorphism. This morphism u is called the *$\mathcal{X}$-envelope* of M. And this envelope is a *monomorphic envelope* if, in addition, u is a monomorphism. Dually, $\mathcal{X}$ is called a *covering class* if for any right R-module M, there exists a homomorphism $p : X(M) \to M$ such that any other homomorphism from an object of $\mathcal{X}$ to M factors through p and moreover, whenever p factors as $p = p \circ h$, h must be an automorphism. This morphism p is called the *$\mathcal{X}$-cover of M* and this cover is said to be an *epimorphic cover* if p is an epimorphism.

A module M having a monomorphic $\mathcal{X}$-envelope $u : M \to X(M)$ (resp. epimorphic $\mathcal{X}$-cover $p : X(M) \to M$) is said to be *invariant under* φ (resp. *coinvariant under* φ), $\varphi : X(M) \to X(M)$, if there exists an endomorphism $f : M \to M$ such that $u \circ f = \varphi \circ u$ (resp. $f \circ p = p \circ \varphi$).

A module M having a monomorphic $\mathcal{X}$-envelope $u : M \to X(M)$ (resp. epimorphic $\mathcal{X}$-cover $p : X(M) \to M$) is said to be *$\mathcal{X}$-automorphism invariant* (resp. *$\mathcal{X}$-automomorphism coinvariant*) if M is invariant (resp. coinvariant) under each automorphism $\varphi : X(M) \to X(M)$.

If a module M is invariant (resp. coinvariant) under each endomorphism $\varphi : X(M) \to X(M)$, then M is called *$\mathcal{X}$-endomorphism invariant* (resp. *$\mathcal{X}$-endomorphism coinvariant*).

When $\mathcal{X}$ is the class of injective modules, $\mathcal{X}$-automorphism invariant modules are usually just called *automorphism-invariant* modules and $\mathcal{X}$-endomorphism invariant modules are called *quasi-injective* modules. When $\mathcal{X}$ is the class of projective modules, $\mathcal{X}$-automorphism coinvariant modules are called *automorphism-coinvariant* modules and $\mathcal{X}$-endomorphism coinvariant modules are called *quasi-projective* modules.

Automorphism-invariant modules have been studied extensively in [1,4,5,8–10,14,16]. On the other hand, it was proved in [5] that a module M is automorphism-invariant if and only if any monomorphism from a submodule N of M to M extends to an endomorphism of M. The main goal of this note is to give a new and more conceptual proof of this result which allows to dualize it to automorphism-coinvariant modules.

Throughout this note, all rings will be associative rings with identity and 'module' will mean a unitary right module unless otherwise stated. We refer to [2,3] for any undefined notion used along the text.

2. Main results

We begin by noting an important structural result from [11] which will be of crucial importance throughout.

Theorem 2.1. *([11]) Let $\mathcal{X}$ be an enveloping (resp., covering) class of modules. If $u : M \to X$ is a monomorphic $\mathcal{X}$-envelope (resp., $p : X \to M$ is an epimorphic cover) of*