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Examples to Birkhoff's quasigroup axioms

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A B S T R A C T

The equational variety of quasigroups is defined by six identities, called Birkhoff's identities. It is known, that only four of them suffice to define the variety; actually, there are nine different combinations of four Birkhoff's identities defining quasigroups, other four combinations define larger varieties and it was open whether the remaining two cases define quasigroups or larger classes. We solve the question here constructing examples of algebras that are not quasigroups and satisfy the open cases of Birkhoff's identities.

1. Introduction

Quasigroups are binary systems $(Q, *)$ such that all equations

$$a * x = b \qquad \text{and} \qquad x * a = b, \qquad \text{for } a, b \in Q$$

have unique solutions. This natural definition, however, has a drawback that subalgebras do need to conserve the existence and homomorphic images the uniqueness of a solution. To deal with this problem, Birkhoff [1] added two more operations / and \ and six axioms

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$$x * (x \backslash y) = y \quad (1) \qquad (y/x) * x = y \quad (2) \qquad x \backslash (x * y) = y \quad (3)$$

$$(y * x)/x = y \quad (4) \qquad x/(y \backslash x) = y \quad (5) \qquad (x/y) \backslash x = y \quad (6),$$

that are true in every quasigroup. Now quasigroups, as a class of algebras in signature $(*, /, \backslash)$, form a variety, that means a class closed for subalgebras, homomorphic images and products.

It turned out quite soon [3] that not all of the six identities are needed, that (1)–(4) suffice since (5) and (6) are consequences of (1)–(4). And it is a natural question: "Which other four-tuples of Birkhoff's identities do also define the entire equational class of quasigroup identities?" Phillips, Pushkashu, Shcherbacov and Shcherbacov [4] studied this question and proved that, among all fifteen four-tuples, there are nine of them defining quasigroups, one defining cancellative left quasigroups, one defining cancellative right quasigroups, one defining divisible left quasigroups and one defining divisible right quasigroups. The form of the equational variety of the remaining two combinations was not discovered and two open problems were thus formulated:

Problem 1. [4] Is a binary algebra $(Q, *, /, \backslash)$ satisfying (1), (2), (5), (6) necessarily a quasigroup? Is a binary algebra $(Q, *, /, \backslash)$ satisfying (3), (4), (5), (6) necessarily a quasigroup?

The answer is negative to both questions, as we see using the following counterexamples:

Example 2. Consider the algebra $(\mathbb{Z}, *, /, \backslash)$ with operations

$$a * b = \left\lfloor \frac{b-a}{2} \right\rfloor, \qquad a/b = b - 2a, \qquad a \backslash b = a + 2b.$$

This algebra satisfies (1), (2), (5), (6) but not (3) and (4).

Example 3. Consider the algebra $(\mathbb{Z}, *, /, \backslash)$ with operations

$$a * b = 2(b-a), \qquad a/b = b - \lfloor a/2 \rfloor, \qquad a \backslash b = a + \lfloor b/2 \rfloor.$$

This algebra satisfies (3), (4), (5), (6) but not (1) and (2).

It is easy to check by hand that both examples have announced properties and therefore we could easily finish our paper here. Nevertheless, it might be interesting to show how the examples were constructed and this is the content of the second section.