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A PI degree theorem for quantum deformations

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A B S T R A C T

We prove that if a filtered quantization A of a finitely generated commutative domain over a field k is a PI algebra, then A is commutative if $\mathrm{char}(k) = 0$, and its PI degree is a power of p if $\mathrm{char}(k) = p$.

1. Introduction

Let F be an algebraically closed field. We show that if a quantum formal deformation A of a commutative domain A_0 over F is a PI algebra, then A is commutative if $\mathrm{char}(F) = 0$, and has PI degree a power of p if $\mathrm{char}(F) = p > 0$. This implies the same result for filtered deformations (i.e., filtered algebras A such that $\mathrm{gr}(A) = A_0$).

Note that a quantum formal deformation of a commutative domain A_0 may fail to be PI, even for finitely generated A_0 in characteristic p (Example 3.3(2)). However, we don't know if this is possible for filtered deformations. Thus we propose

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Question 1.1. Let $\mathrm{char}(F) = p > 0$, and A be a filtered deformation of a commutative finitely generated domain A_0 over F. Must A be a PI algebra? In other words, must the division ring of quotients of A be a central simple algebra?

This question is closely related to the question asked in the introduction to [2], which would have affirmative answer if the answer to Question 1.1 is affirmative. We don't know the answer to either of these questions even when A_0 is a polynomial algebra with generators in positive degrees.

2. Deformations of fields

Let F be an algebraically closed field, and A_0 a field extension of F. Let A be a quantum formal deformation of A_0 over $F[[\hbar]]$, i.e. an $F[[\hbar]]$-algebra isomorphic to $A_0[[\hbar]]$ as an $F[[\hbar]]$ module and equipped with an isomorphism of algebras $A/(\hbar) \cong A_0$ (for basics and notation on deformations, see [3], Section 2).

Theorem 2.1. *Suppose that A is a PI algebra of degree d.*

(i) If $\mathrm{char} F = 0$, then $d = 1$ (i.e., A is commutative).

(ii) If $\mathrm{char} F = p > 0$, then d is a power of p.

Proof. Let C be the center of A. It is easy to see that the division algebra of quotients of A is $A[\hbar^{-1}]$ with center $C[\hbar^{-1}]$ (see [3], Example 2.7). Moreover, by Posner's theorem ([5], 13.6.5), $A[\hbar^{-1}]$ is a central division algebra over $C[\hbar^{-1}]$ of degree d, so $[A[\hbar^{-1}] : C[\hbar^{-1}]] = d^2$.

Let $C_0 = C/(\hbar)$. It is clear that C_0 is a subfield of A_0, and C is a (commutative) formal deformation of C_0.

Lemma 2.2. $[A_0 : C_0] = d^2$.

Proof. Let $a_1^0, ..., a_m^0 \in A_0$ be linearly independent over C_0. Let $a_1, ..., a_m$ be lifts of these elements to A. Then $a_1, ..., a_m$ are linearly independent over C and hence over $C[\hbar^{-1}]$. Thus $m \leq d^2$. Moreover, if $a_1^0, ..., a_m^0$ are a basis of A_0 over C_0 then $a_1, ..., a_m$ are a free basis of A over C and hence a basis of $A[\hbar^{-1}]$ over $C[\hbar^{-1}]$, so $m = d^2$. □

Now for every integer $r \geq 0$, let $A_r \subset A_0$ be the field of all $x \in A_0$ which admit a lift to a central element of $A/(\hbar^{r+1})$. Note that $A_r \supset A_{r+1}$, and by Lemma 2.2, this is a finite field extension.

Let us now prove (i). Assume the contrary, i.e. that A is noncommutative. Let r be the largest integer such that $[a, b] \in \hbar^r A$ for all $a, b \in A$. Then we have a nonzero Poisson bracket on A_0 given by $\{a_0, b_0\} = \hbar^{-r}[a, b] \bmod \hbar$, where a, b are any lifts of a_0, b_0 to A. Moreover, by definition $\{,\}$ is bilinear over A_r. Recall that $\{,\}$ is a derivation in each argument, and that any K-linear derivation of a finite extension of a field K of charac-