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On the utility of Robinson–Amitsur ultrafilters. II

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ABSTRACT

We extend the result of the previous paper under the same title about embedding of ideal-determined algebraic systems into ultraproducts, to arbitrary algebraic systems, and to ultraproducts over κ-complete ultrafilters. We also discuss the scope of applicability of this result, and correct a mistake from the previous paper concerning homomorphisms from ultraproducts.

0. Introduction

In [8], an old and simple trick, used by A. Robinson and S. Amitsur (see [1, Proof of Theorem 15] and [2, Theorem 3]) to establish, in the context of ring theory, an embedding of certain rings in ultraproducts, was generalized to algebraic systems with ideal-determined congruences. Here we push it to the full generality, for arbitrary algebraic systems without any restrictions on their congruences.

1. Recollection on congruences, ultrafilters, and ultraproducts

We refer to [3] and [6] for the rudiments of universal algebra and model theory we need.

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Let us recall some basic notions and fix notation. We consider the most general algebraic systems, i.e. sets with a number of operations Ω defined on them, of, generally, various arity. The signature Ω is arbitrary, but fixed (so, in what follows all algebraic systems are supposed to be of signature Ω). The set of all congruences on a given algebraic system A forms a partially ordered set (actually, a lattice), and the minimal element in this set is the *trivial congruence*, coinciding with the diagonal $\{(a,a) \mid a \in A\}$ in $A \times A$. The intersection of all congruences containing a given relation ρ, i.e. a set of pairs of elements from $A \times A$, is called a *congruence generated by* ρ and is denoted by $\mathrm{Con}(\rho)$. In the case of one-element relation, i.e. a single pair $(a,b) \in A \times A$ with $a \neq b$, we shorten this notation to $\mathrm{Con}(a,b)$ and speak about *principal congruences*.

Given a cardinal $\kappa > 2$, let us call a set $\mathscr{S}$ of sets *κ-complete*, if the intersection of any nonempty set of fewer than κ elements of $\mathscr{S}$ belongs to $\mathscr{S}$. If a set $\mathscr{S}$ of subsets of a set $\mathbb{I}$ satisfies a weaker condition – that the intersection of any nonempty set of fewer than κ elements of $\mathscr{S}$ contains an element of $\mathscr{S}$ – then the set of subsets of $\mathbb{I}$ which are oversets of all such intersections is a κ-complete filter on $\mathbb{I}$ containing $\mathscr{S}$ (this is an obvious generalization of the standard and frequently employed fact that any set of subsets satisfying the finite intersection property can be extended to a filter).

An algebraic system A is called *κ-subdirectly irreducible*, if either of the following two equivalent conditions is satisfied:

- The set of nontrivial congruences of A is κ-complete.
- If A embeds in the direct product of $< \kappa$ algebraic systems $\prod_{i \in \mathbb{I}} B_i$, $|\mathbb{I}| < \kappa$, then A embeds in one of B_i's.

Obviously, the condition of ω-subdirect irreducibility, also called *finite subdirect irreducibility*, is equivalent to n-subdirect irreducibility for any finite $n > 2$. Note that ω-complete filters (ultrafilters) are just the usual filters (ultrafilters).

Given a filter (respectively, ultrafilter) $\mathscr{F}$ on a set $\mathbb{I}$, the quotient of the direct product of algebraic systems $\prod_{i \in \mathbb{I}} A_i$ by the congruence

$$\theta_{\mathscr{F}} = \{(a,b) \in (\prod_{i \in \mathbb{I}} A_i) \times (\prod_{i \in \mathbb{I}} A_i) \mid \{i \in \mathbb{I} \mid a(i) = b(i)\} \in \mathscr{F}\}$$

is called *filtered product* (respectively, *ultraproduct*) of the corresponding algebraic systems, and is denoted by $\prod_{\mathscr{F}} A_i$.

As any κ-complete ultrafilter on a set of cardinality $< \kappa$ is principal, the condition of κ-subdirect irreducibility of an algebraic system A may be trivially reformulated as follows:

- If A embeds in the direct product of $< \kappa$ algebraic systems $\prod_{i \in \mathbb{I}} B_i$, $|\mathbb{I}| < \kappa$, then A embeds in the ultraproduct $\prod_{\mathscr{U}} B_i$ for some ultrafilter $\mathscr{U}$ on $\mathbb{I}$.