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One-tilting classes and modules over commutative rings

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A R T I C L E I N F O

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A B S T R A C T

We classify 1-tilting classes over an arbitrary commutative ring. As a consequence, we classify all resolving subcategories of finitely presented modules of projective dimension at most 1. Both these collections are in 1-1 correspondence with faithful Gabriel topologies of finite type, or equivalently, with Thomason subsets of the spectrum avoiding a set of primes associated in a specific way to the ring. We also provide a generalization of the classical Fuchs and Salce tilting modules, and classify the equivalence classes of all 1-tilting modules. Finally we characterize the cases when tilting modules arise from perfect localizations.

1. Introduction

The classification of tilting classes and modules was done gradually, starting with abelian groups ([18]), then small Dedekind domains, first assuming V = L ([29,30]), and then in ZFC ([8]), for Prüfer domains ([6]), and almost perfect domains ([5]). Recently,

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in [4] the authors classified tilting classes of a commutative noetherian ring in terms of finite sequences of subsets of the Zariski spectrum of R. In particular, they proved that 1-tilting classes correspond bijectively to specialization closed subsets of $\mathrm{Spec}(R)$ that do not contain associated primes of R. We generalize this result to arbitrary commutative rings by showing that there is a one-to-one correspondence between 1-tilting classes and Thomason subsets of $\mathrm{Spec}(R)$ that avoid primes "associated" to R in certain sense. Thomason subsets of the spectrum coincide with specialization closed subsets in the noetherian case, and seem to be the correct generalization in various classification theorems. The prime example of this phenomenon is the classification of compactly generated localizing subcategories of the unbounded derived category of R done first by Neeman for noetherian rings and then in general by Thomason ([28]).

As in the noetherian case in ([4]), we start working in the dual setting of cotilting classes. Even though there is an explicit duality between tilting modules and cotilting modules of cofinite type, the one way nature of the duality makes the tilting side harder to approach. For example, cotilting modules over commutative noetherian case are described in [27], but tilting modules were described only for special classes of noetherian rings. The crucial step in our approach is to show that a 1-cotilting class is of cofinite type if and only if it is closed under injective envelopes (Corollary 3.13).

Alternatively, 1-tilting classes over a commutative ring R correspond bijectively to faithful finitely generated Gabriel topologies over R. From this point of view, our classification generalizes directly results for Prüfer domains from [6]. If R is not semihereditary, one has to replace the cyclic generators of the hereditary torsion class by their Auslander–Bridger transposes in order to describe the resolving subcategories of finitely presented modules of projective dimension at most 1. In the second part of the paper, we use this idea and construct an associated tilting module for each 1-tilting class over a commutative ring. This construction generalizes the Fuchs and Salce tilting modules introduced by Facchini, Fuchs–Salce, and Salce ([16,17,25]) from multiplicative sets over a domain and finitely generated Gabriel topology over a Prüfer domain to general faithful finitely generated Gabriel topology over a commutative ring.

In the rest of the second section we use the "minimality" of the constructed 1-tilting modules and provide an elementary proof of the commutative version of the recently solved Saorín's problem ([10]). Finally, in the last section we show that a 1-tilting module arises from a perfect localization if and only if the associated Gabriel topology is perfect and the induced perfect localization has projective dimension 1.

2. Preliminaries

2.1. Basic notation and cotorsion pairs

Given an (associative, unital) ring R, we denote by Mod-R the category of all right R-modules and by mod-R the full subcategory of Mod-R consisting of all finitely presented right R-modules.