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# Groups in which every non-abelian subgroup is self-centralizing

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We study groups having the property that every non-abelian subgroup contains its centralizer. We describe various classes of infinite groups in this class, and address a problem of Berkovich regarding the classification of finite $p$-groups with the above property.



## 1. Introduction

A subgroup $H$ of a group $G$ is *self-centralizing* if the centralizer $C_G(H)$ is contained in $H$. Clearly, an abelian subgroup $A$ of $G$ is self-centralizing if and only if $C_G(A) = A$. In particular, the trivial subgroup of $G$ is self-centralizing if and only if $G$ is trivial.

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The structure of groups in which many non-trivial subgroups are self-centralizing has been studied in several papers. In [5] it has been proved that a locally graded group (that is, a group in which every non-trivial finitely generated subgroup has a proper subgroup of finite index) in which all non-trivial subgroups are self-centralizing has to be finite; therefore it has to be cyclic of prime order or a non-abelian group whose order is a product of two different primes. The papers [5] and [6] deal with the class $\mathfrak{C}$ of groups in which every non-cyclic subgroup is self-centralizing; in particular, a complete classification of locally finite $\mathfrak{C}$-groups is given.

In this paper, we study the class $\mathfrak{A}$ of groups in which every non-abelian subgroup is self-centralizing. We note that the class $\mathfrak{A}$ is fairly wide. Clearly, it contains all $\mathfrak{C}$-groups. It also contains the class of commutative-transitive groups (that is, groups in which the centralizer of each non-trivial element is abelian), see [13]. Moreover, by definition, the class $\mathfrak{A}$ contains all minimal non-abelian groups (that is, non-abelian groups in which every proper subgroup is abelian); in particular, Tarski monsters are $\mathfrak{A}$-groups.

The structure and main results of the paper are as follows. In Section 2 we derive some basic properties of $\mathfrak{A}$-groups; these results are crucial for the further investigations in the subsequent sections. In Section 3 we consider infinite nilpotent $\mathfrak{A}$-groups; for example, we prove that such groups are abelian, which reduces the investigation of nilpotent $\mathfrak{A}$-groups to finite $p$-groups in $\mathfrak{A}$. Infinite supersoluble groups in $\mathfrak{A}$ are classified in Section 4; for example, we prove that if such a group has no element of order 2, then it must be abelian. In Section 5 we discuss some properties of soluble groups in $\mathfrak{A}$. Lastly, in Section 6, we consider finite $\mathfrak{A}$-groups, and we derive various characterizations of finite groups in $\mathfrak{A}$. Motivated by Section 3, we focus on finite $p$-groups in $\mathfrak{A}$; Problem 9 of [1] asks for a classification of such groups. This appears to be hard, as there seem to be many classes of finite $p$-groups that belong to $\mathfrak{A}$. We show that all finite metacyclic $p$-groups are in $\mathfrak{A}$, and classify the finite $p$-groups in $\mathfrak{A}$ which have maximal class or exponent $p$.

## 2. Basic properties of $\mathfrak{A}$-groups

We collect some basic properties of $\mathfrak{A}$-groups. Since every free group lies in $\mathfrak{A}$, the class $\mathfrak{A}$ is not quotient closed. On the other hand, $\mathfrak{A}$ obviously is subgroup closed. Similarly, the next lemma is an easy observation.

**Lemma 2.1.** *If $G$ is an $\mathfrak{A}$-group, then its center $Z(G)$ is contained in every non-abelian subgroup of $G$.*

As usual, we denote by $\Phi(G)$ the Frattini subgroup of a group $G$.

**Lemma 2.2.** *If $G \in \mathfrak{A}$ is non-abelian group, then $Z(G) \leq \Phi(G)$.*

**Proof.** Let $M$ be a maximal subgroup of $G$. If $M$ is abelian and $Z(G) \not\leq M$, then $MZ(G) = G$, hence $G$ is abelian, a contradiction. On the other hand, if $M$ is non-abelian,