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Quantization of special symplectic nilpotent orbits and normality of their closures

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ABSTRACT

We study the regular function ring $R(\mathcal{O})$ for all symplectic nilpotent orbits $\mathcal{O}$ with even column sizes. We begin by recalling the quantization model for all such orbits by Barbasch using unipotent representations. With this model, we express the multiplicities of fundamental representations appearing in $R(\mathcal{O})$ by a parabolically induced module. Finally, we will use this formula to give a criterion on the normality of the Zariski closure $\overline{\mathcal{O}}$ of $\mathcal{O}$.

1. Introduction

Let $G = Sp(2n, \mathbb{C})$ be the complex symplectic group. The G-conjugates of a nilpotent element $X \in \mathfrak{g}$ form a **nilpotent orbit** $\mathcal{O} \subset \mathfrak{g}$. The idea of the orbit method, first proposed by Kirillov, suggests that one can 'attach' a unitary representation to each nilpotent orbit $\mathcal{O}$. In [21], the author used tools from unipotent representations and dual pair correspondence to achieve this goal for spherical, special nilpotent orbits and their covers (see Theorem A of [21]). In the following work, we would like to study a larger class of nilpotent orbits.

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It is well-known that all such nilpotent orbits are parameterized by partitions, where the partition corresponds to the size of the Jordan blocks. For $Sp(2n, \mathbb{C})$, nilpotent orbits are identified with the partitions of $2n$ in which odd parts occur with even multiplicity.

In fact it is sometimes more convenient to look at the column sizes, or the **dual partition**, of a given partition. More precisely, let $\psi = [r_1, r_2, \dots, r_i]$ be a partition of n, with $r_1 \geq r_2 \geq \dots \geq r_i > 0$, its dual partition is given by $\psi^* = (c_k, c_{k-1}, \dots, c_1)$, where $c_{k+1-j} = \#\{i | r_i \geq j\}$. We will use square bracket $[r_1, r_2, \dots]$ to denote the partition of a nilpotent orbit, and round bracket $(c_k, c_{k-1}, \dots)$ to denote the dual partition of the same orbit.

Given two partitions $\varsigma = [r_1, \dots, r_p]$, $\psi = [r'_1, \dots, r'_q]$, we define their **union** $\varsigma \cup \psi = [s_1, \dots, s_{p+q}]$, where $\{s_1 \dots, s_{p+q}\} = \{r_1, \dots, r_p, r'_1, \dots, r'_q\}$ as sets and $s_1 \geq s_2 \geq \dots \geq s_{p+q}$. Also, define the **join** to be $\varsigma \vee \psi = (\varsigma^* \cup \psi^*)^*$, so that if $\varsigma = (c_m, \dots, c_0)$, $\psi = (c'_n, \dots, c'_0)$, then $\varsigma \vee \psi = (d_{m+n+1}, \dots, d_0)$, where $\{d_{m+n+1} \dots, d_0\} = \{c_m, \dots, c_0, c'_n, \dots, c'_0\}$ as sets and $d_{m+n+1} \geq d_{m+n} \geq \dots \geq d_0$.

Here is a restatement of the characterization of nilpotent orbits for $Sp(2n, \mathbb{C})$, which is implicit in the construction of nilpotent orbit closures in [10]: Any nilpotent orbit for $Sp(2n, \mathbb{C})$ can be parameterized by a partition of $2n$ with column sizes $(c_{2k}, c_{2k-1}, \dots, c_0)$, where $c_{2k} \geq c_{2k-1} \geq \dots \geq c_0 \geq 0$ (by insisting c_{2k} to be the longest column, we put $c_0 = 0$ if necessary), such that $c_{2i} + c_{2i-1}$ is even for all $i \geq 0$ (where $c_{-1} = 0$).

For most parts of the following work, we study the rings of regular functions $R(\mathcal{O})$ of nilpotent orbits ([13,14]). More precisely, we study symplectic nilpotent orbits of the form $\mathcal{O} = (2a_{2k}, \dots, 2a_1, 2a_0)$, i.e. all columns of $\mathcal{O}$ are even. In this setting, the following is known to be true:

Theorem 1.1 *(Barbasch – [6] and [7], p. 29). Let $\mathcal{O} = (2a_{2k}, \dots, 2a_1, 2a_0)$ be a nilpotent orbit such that $a_{2i-1} > a_{2i-2}$ for all i. Then as $G \cong K_{\mathbb{C}}$-modules, the spherical unipotent representation X_{triv} attached to $\mathcal{O}$ satisfies*

$$X_{\text{triv}} \cong R(\mathcal{O}).$$

In fact, Barbasch in [6] proved a much more general statement than Theorem 1.1 for other classical Lie groups and other unipotent representations. More specifically, the other unipotent representations X_π (see Equation (2)) correspond to the global sections of some G-equivariant vector bundle $G \times_{G_e} V_\pi$ of $\mathcal{O}$. This essentially verifies a conjecture of Vogan (Conjecture 12.1 of [17]) for such orbits. More details are given in Remark 3.3.

With Theorem 1.1, we can essentially compute the multiplicity of any irreducible representations appearing in $R(\mathcal{O})$. In particular, we focus on the fundamental representations of $G = Sp(2n, \mathbb{C})$, given by

$$\mu_i := \wedge^i \mathbb{C}^{2n} / \wedge^{i-2} \mathbb{C}^{2n}$$

for $i = 1, 2, \dots, n$ (if $i - 2 < 0$, take $\wedge^{i-2}\mathbb{C}^{2n} = \text{triv}$). We have the following formula for the multiplicities of fundamental representations for a larger class of nilpotent orbits than in Theorem 1.1: