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On covariants in exterior algebras for the even special orthogonal group

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ABSTRACT

Let $G := SO(2n, \mathbb{C})$ be the even special orthogonal group and let M_{2n}^{+} (resp. M_{2n}^{-}) be the space of symmetric (resp. skew-symmetric) complex matrices with respect to the usual transposition.
We study the structure of $B^{+} := \left(\bigwedge(M_{2n}^{+})^{*} \otimes M_{2n}^{-}\right)^{G}$, the space of G-equivariant skew-symmetric matrix valued alternating multilinear maps on the space of symmetric n-tuples of matrices, with G acting by conjugation.
Further, we decompose B as the direct sum $B \simeq B^{+} \oplus B^{-}$, where $B^{\mp} := \left(\bigwedge(M_{2n}^{\bullet})^{*} \otimes M_{2n}^{\pm}\right)^{G}$.
We prove that B^{+} is a free module over a certain subalgebra of invariants $A := \left(\bigwedge(M_{2n}^{+})^{*}\right)^{G}$ of rank $2n$. We give an explicit description for the basis of this module. Furthermore we prove new trace polynomial identities for symmetric matrices.
Finally we show, using a computer assisted computation made with the LiE software, that $B^{-} := \left(\bigwedge(M_{2n}^{+})^{*} \otimes M_{2n}^{+}\right)^{G}$ doesn't satisfy a similar property.

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Introduction

The problem of studying the irreducible components of the exterior algebra of a complex simple Lie algebra is a classical topic in representation theory (see for example [5]). A precise description of very special cases, like the adjoint representation [3] or the little adjoint representations [2], has been recently obtained. These modules of covariants have properties of freeness over suitable subalgebras of invariants. Furthermore, the results on the adjoint case in [3] can be stated and generalized to the context of infinitesimal symmetric spaces.

Let $\mathfrak{g}$ be a complex finite dimensional semisimple Lie algebra and $\sigma : \mathfrak{g} \to \mathfrak{g}$ an indecomposable involution. Then we can classically decompose $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of σ-fixed points and $\mathfrak{p}$ the eigenspace relative to -1. We denote the infinitesimal symmetric space associated to $\mathfrak{p}$ by the pair $(\mathfrak{g}, \mathfrak{k})$. Let us remark that the adjoint action of $\mathfrak{g}$ induces a natural action of $\mathfrak{k}$ on $\mathfrak{p}$ and consequently on $\bigwedge \mathfrak{p}^*$.

Then we have the following result:

Theorem 0.1. *([3,4]) With the previous notation, let $(\mathfrak{g}, \mathfrak{k})$ be an infinitesimal symmetric space such that $(\bigwedge \mathfrak{p}^*)^{\mathfrak{k}}$ is an exterior algebra of type $\wedge(x_1, ..., x_r)$, where the x_i's are ordered by their degree and $r := rk(\mathfrak{g}) - rk(\mathfrak{k})$. Then the algebra $B := (\bigwedge \mathfrak{p}^* \otimes \mathfrak{g})^{\mathfrak{k}}$ is a free module of rank $4r$ on the subalgebra of $(\bigwedge \mathfrak{p}^*)^{\mathfrak{k}}$ generated by the elements $x_1, ..., x_{r-1}$.*

Despite being based on different approaches, the above-mentioned works rely on the property of the invariants of being an exterior algebra to deduce freeness properties.

Let us recall that, by the Hopf–Koszul–Samelson Theorem, the invariants $(\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$ for a complex simple Lie algebra $\mathfrak{g}$ are always an exterior algebra in $rk(\mathfrak{g})$ generators of predictable degree. And apart from this case, in which we consider $\mathfrak{g}$ as the infinitesimal symmetric space associated with the pair $(\mathfrak{g} \oplus \mathfrak{g}, \Delta(\mathfrak{g}))$, with $\Delta(\mathfrak{g})$ being the diagonal copy of $\mathfrak{g}$ in $\mathfrak{g} \oplus \mathfrak{g}$, there are very few others with this property. They are in fact the pairs $(\mathfrak{sl}(2n), \mathfrak{sp}(2n))$, $(\mathfrak{sl}(2n+1), \mathfrak{so}(2n+1))$, $(\mathfrak{so}(2n), \mathfrak{so}(2n-1))$, $(\mathfrak{e}_6, \mathfrak{f}_4)$.

In this paper we present an example of an infinitesimal symmetric space, associated to the pair $(\mathfrak{sl}(2n), \mathfrak{so}(2n))$, such that the invariants are not an exterior algebra, nevertheless we can deduce some kind of freeness property for covariants.

In fact we study the structure of a more general space, $B := \left(\bigwedge (M_{2n}^+)^* \otimes M_{2n}\right)^G$, which could be seen as the space of the G-equivariant matrix valued alternating multilinear maps on the space of symmetric n-tuples of matrices, with $G = SO(2n, \mathbb{C})$ the even special orthogonal group acting by conjugation.

We put on B a structure of algebra defining a skew-symmetric product induced by the matrix product and then we see B as a module over the algebra of invariants $A := \left(\bigwedge (M_{2n}^+)\right)^G$ in a natural way. We give a precise description of both these structures.

Furthermore, it is convenient for our goals to decompose B as the direct sum $B \simeq B^+ \oplus B^-$, where $B^{\mp} := \left(\bigwedge (M_{2n}^{\bullet})^* \otimes M_{2n}^{\pm}\right)^G$.