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Infinite-dimensional cohomology of $\mathbf{SL_2}\left(\mathbb{Z}[t,t^{-1}]\right)$

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A B S T R A C T

For J an integral domain and F its field of fractions, we construct a map from the 3-skeleton of the classifying space for $\Gamma = \mathbf{SL_2}(J[t,t^{-1}])$ to a Euclidean building on which Γ acts. We then find an infinite family of independent cocycles in the building and lift them to the classifying space, thus proving that the cohomology group $H^2\left(\mathbf{SL_2}\left(J[t,t^{-1}]\right);F\right)$ is infinite-dimensional.

1. Introduction

Let J be an integral domain with identity and F its field of fractions. Our goal in this paper is to prove the following theorem:

Theorem 1. *$H^2\left(\mathbf{SL_2}\left(J[t,t^{-1}]\right);F\right)$ is infinite-dimensional.*

This theorem generalizes several earlier results about the finiteness properties of $\mathbf{SL_2}\left(\mathbb{Z}[t,t^{-1}]\right)$. In [5], Krstić–McCool prove that $\mathbf{SL_2}(J[t,t^{-1}])$, among other related groups, is not F_2.

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In [2], Bux–Wortman use geometric methods to prove that $\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}])$ is also not FP_2. In particular, they use the action of $\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}])$ on a product of locally infinite trees. Bux–Wortman also ask whether their proof can be extended to show the stronger result that $H_2(\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}]);\mathbb{Z})$ is infinitely generated. Knudson proves that this is the case in [6] using algebraic methods. Note that this result also follows from Theorem 1: Since $\mathbb{Q}$ is a field,

$$H^2(\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}]);\mathbb{Q}) \simeq \mathrm{Hom}(H_2(\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}]);\mathbb{Q}),\mathbb{Q})$$

Since $\mathrm{Hom}(H_2(\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}]);\mathbb{Q}),\mathbb{Q})$ is infinite dimensional, so is $H_2(\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}]);\mathbb{Q})$, which implies that $H_2(\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}]);\mathbb{Z})$ is not finitely generated as a $\mathbb{Z}$-module.

The methods in this paper will be geometric. We will define two spaces on which $\mathbf{SL_2}(J[t,t^{-1}])$ acts: one a Euclidean building as in [2], and the other a classifying space for $\mathbf{SL_2}(J[t,t^{-1}])$. A map between these spaces will allow us to explicitly define an infinite family of independent cocycles in $H^2(\mathbf{SL_2}(J[t,t^{-1}]);F)$.

The methods used are based on those of Cesa–Kelly in [4], where they are used to show that $H^2(\mathbf{SL_3}(\mathbb{Z}[t]);\mathbb{Q})$ is infinite-dimensional. Wortman follows a similar outline in [7].

2. The Euclidean building

Throughout, let J be an integral domain with identity, F the field of fractions over J, and $\Gamma = \mathbf{SL_2}(J[t,t^{-1}])$.

We begin by recalling the structure of a Euclidean building on which Γ acts. The construction and notation follow Bux–Wortman in [2]. Let ν_∞ and ν_0 be the valuations on $F(t)$ giving multiplicity of zeros at infinity and at zero, respectively. More precisely, $\nu_\infty\left(\frac{p(t)}{q(t)}\right) = \deg(q(t)) - \deg(p(t))$, and $\nu_0\left(\frac{r(t)}{s(t)}t^n\right) = n$, where t does not divide the polynomials r and s. Let T_∞ and T_0 be the Bruhat–Tits trees associated to $\mathbf{SL_2}(F(t))$ with the valuations ν_∞ and ν_0, respectively. We will consider each tree as a metric space with edges having length 1. Let $X = T_\infty \times T_0$.

Since $F((t^{-1}))$ (respectively $F((t)))$ is the completion of $F(t)$ with respect to ν_∞ (resp. ν_0), $\mathbf{SL_2}(F((t^{-1})))$ (resp. $\mathbf{SL_2}(F(t)))$ acts on the tree T_∞ (resp. T_0). Therefore the group $\mathbf{SL_2}(F((t^{-1}))) \times \mathbf{SL_2}(F((t)))$ acts on X.

Throughout this paper, we will regard Γ and $\mathbf{SL_2}(F(t))$ as diagonal subgroups of $\mathbf{SL_2}(F((t^{-1}))) \times \mathbf{SL_2}(F((t)))$, which act on X via that embedding.

Let L_∞ (resp. L_0) be the unique geodesic line in T_∞ (resp. T_0) stabilized by the diagonal subgroup of $\mathbf{SL_2}(F(t))$. Let $\ell_\infty : \mathbb{R} \longrightarrow L_\infty$ (resp. $\ell_0 : \mathbb{R} \longrightarrow L_0$) be an isometry with $\ell_\infty(0)$ (resp. $\ell_0(0)$) the unique vertex with stabilizer $\mathbf{SL_2}(F[t^{-1}])$ (resp. $\mathbf{SL_2}(F[t])$). Let $x_0 = (\ell_\infty(0), \ell_0(0))$ serve as a basepoint of X and $\Sigma = L_\infty \times L_0$ so that Σ is an apartment of X.