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# The weak Lefschetz property for monomial ideals of small type ☆

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ABSTRACT

In this work a combinatorial approach towards the weak Lefschetz property is developed that relates this property to enumerations of signed perfect matchings as well as to enumerations of signed families of non-intersecting lattice paths in certain triangular regions. This connection is used to study Artinian quotients by monomial ideals of a three-dimensional polynomial ring. Extending a main result in the recent memoir [1], we completely classify the quotients of type two that have the weak Lefschetz property in characteristic zero. We also derive results in positive characteristic for quotients whose type is at most two.



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## 1. Introduction

A standard graded Artinian algebra $A$ over a field $K$ is said to have the *weak Lefschetz property* if there is a linear form $\ell \in A$ such that the multiplication map $\times\ell : [A]_i \to [A]_{i+1}$ has maximal rank for all $i$ (i.e., it is injective or surjective). The algebra $A$ has the *strong Lefschetz property* if $\times\ell^d : [A]_i \to [A]_{i+d}$ has maximal rank for all $i$ and $d$. The names are motivated by the conclusion of the Hard Lefschetz Theorem on the cohomology ring of a compact Kähler manifold. Many algebras are expected to have the Lefschetz properties. However, deciding whether a Lefschetz property does indeed hold true is often a very challenging problem.

The presence of the weak Lefschetz property has profound consequences for an algebra (see [19]). For example, Stanley used this in his contribution [37] towards the proof of the so-called $g$-Theorem that characterizes the face vectors of simplicial polytopes. It has been a longstanding conjecture whether this characterization extends to the face vectors of all triangulations of a sphere. In fact, this would be one of the consequences if one can show the so-called algebraic $g$-Conjecture, which posits that a certain algebra has the strong Lefschetz property (see [31] and [32]). Although there has been a flurry of papers studying the Lefschetz properties in the last decade (see, e.g., [2–5,13,17,18,21–23,25,28]), we currently seem far from being able to decide the above conjectures. Indeed, the need for new methods has led us to consider lozenge tilings, perfect matchings, and families of non-intersecting lattice paths. We use this approach to establish new results about the presence or the absence of the weak Lefschetz property for quotients of a polynomial ring $R$ in three variables. This is the first open case as any Artinian quotient of a polynomial ring in two variables has even the strong Lefschetz property in characteristic zero [19].

If $I$ is a monomial ideal, then $R/I$ is Artinian of type one if and only if $I$ is generated by the powers of the three variables. It is well-known that in this case $R/I$ has the Lefschetz properties if the base field has characteristic zero (see [36,35,38,12]). We extent this result by providing a version for base fields of arbitrary characteristic (see Theorem 6.2).

Monomial algebras $R/I$ of type two were considered in the recent memoir [1]. One of its main results says that, in characteristic zero, these algebras have the weak Lefschetz property, provided they are also level. Examples show that this may fail if one drops the level assumption or if $K$ has positive characteristic. However, the intricate proof in the level case of [1, Theorem 6.2] does not give any insight when such failures occur. We resolve this by completely classifying all type two algebras that have the weak Lefschetz property if the characteristic is zero or large enough (see Theorem 7.2 and Proposition 7.9).

The structure of this paper is as follows. In Section 2, we recall or derive some general results about the presence of the weak Lefschetz property. In Section 3, we describe a key relation between a monomial ideal and a triangular region, a certain planar region, as introduced in [11]. In Section 4, we consider signed lozenge tilings of a triangular region, using two a priori different signs. We show that both signs lead to enumerations of signed lozenge tilings that completely control the presence of the weak Lefschetz