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Diophantine equations via cluster transformations

Philipp Lampe

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ABSTRACT

Motivated by Fomin–Zelevinsky's theory of cluster algebras we introduce a variant of the Markov equation; we show that all natural solutions of the equation arise from an initial solution by cluster transformations.

Contents

E-mail address: lampe@math.uni-bielefeld.de.

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1. Cluster algebras

1.1. Introduction

Fomin and Zelevinsky's cluster theory provides a common combinatorial framework for problems in representation theory, Lie theory, hyperbolic geometry and mathematical physics. The theory was initiated in a series of four influential papers [11,12,2,13]. The key notion to define a cluster algebra is the so-called *mutation*, which we will recall in the next section. We can mutate *quivers* and *cluster variables*. Surprisingly, cluster algebra mutations describe interesting phenomena in various branches of mathematics. For example, a correspondence between non-initial cluster variables and positive roots in a certain root system establishes a link to Lie theory. There are deeper links to Lie theory. In fact, a conjectural correspondence between cluster monomials and Lusztig's canonical basis elements was one of the original motivations to introduce cluster algebras. For a different example, the Caldero–Chapoton map [4] links cluster algebras with quiver representations. In this context, mutation is related to tilting. Cluster algebras also occur in hyperbolic geometry, where mutation is related to Ptolemy's theorem. Some cluster algebras admit invariants of mutation, which play an important role in the context of dynamical systems.

Markov's equation provides a common number-theoretic framework for problems in representation theory, geometry and arithmetic. The key notion to solve the Markov equation is sometimes called *Vieta jumping*. We will recall it in Section 2.1. Although we can solve the Markov equation by elementary methods, it describes interesting phenomena in various branches in mathematics. For example, Gorodentsev–Rudakov [16] show that the solutions of the Markov equation describe ranks of exceptional vector bundles on projective spaces.

In this article, we wish to show how cluster mutations can generate all solutions of a particular Diophantine equation. The form of the Diophantine equation is related to a Laurent polynomial in an upper cluster algebra, which is invariant under mutation. In Section 1.2 we recall the basic notions of cluster theory. Especially, we give a formal definition of the aforementioned mutation process and present its main features. In the following Section 1.3 we introduce Fomin–Shapiro-Thurston's surface cluster algebras [10]. The construction is based on work of Fock–Goncharov [6,7] and Gekhtman–Shapiro–Vainshtein [17]. Surface cluster algebras provide interesting instances of cluster algebras with invariants of mutation. Section 2.1 introduces and solves Markov's equation and interprets it from a cluster theoretic point of view. We introduce a new Diophantine equation in Section 2.2 and show that we can solve the equation by cluster transformations. In Section 2.3 we study another cluster algebra with an invariant and ask some questions about the corresponding Diophantine equation.

1.2. Rudiments of cluster algebras

Let us briefly recall the definition of a cluster algebra. For the rest of the section, we fix a positive integer $n \in \mathbb{N}$. For brevity, we write $[n]$ instead of $\{1, 2, \dots, n\}$.