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Geometric aspects of Iterated Matrix Multiplication

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ABSTRACT

This paper studies geometric properties of the Iterated Matrix Multiplication polynomial and the hypersurface that it defines. We focus on geometric aspects that may be relevant for complexity theory such as the symmetry group of the polynomial, the dual variety and the Jacobian loci of the hypersurface, that are computed with the aid of representation theory of quivers.

1. Introduction

Let q be a positive integer and let Mat_q denote the vector space of $q \times q$ matrices with complex coefficients. For a positive integer n, we denote by IMM_q^n the Iterated Matrix Multiplication polynomial, that is the polynomial on the vector space $V := Mat_q^{\oplus n}$ of n-tuples of $q \times q$ matrices whose value on the n-tuple of matrices $(X_1, \dots, X_n)$ is $\mathrm{trace}(X_n \cdots X_1)$. Thus, IMM_q^n is a homogeneous polynomial of degree n in nq^2 variables.

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The main motivation for this study is the completeness of particular instances of IMM_q^n for some complexity classes. For $q = 3$, the sequence of polynomials IMM_3^n is $\mathbf{VP}_e$-complete [5]; $\mathbf{VP}_e$ is the complexity class of sequences of polynomials that admit a *small formula* (see e.g. [16, Ch. 13] for details). For $q = n$, the sequence of polynomials IMM_n^n is $\mathbf{VQP}$-complete [8]; $\mathbf{VQP}$ is the same complexity class for which the determinant polynomial $\det_n$ is complete; moreover, $\mathbf{VQP}$ is equivalent to polynomially sized algebraic branching programs (see e.g. [14] and [11]).

We use tools from algebraic geometry and representation theory in order to study geometric properties of the polynomial IMM_q^n. We determine the symmetry group of the polynomial IMM_q^n and we prove that this polynomial is characterized by its symmetry group. We make a study of geometric properties of the algebraic hypersurface $\mathcal{I}mm_q^n \subseteq Mat_q^{\oplus n} \simeq \mathbb{C}^{nq^2}$ cut out by the polynomial IMM_q^n: we determine the dimension of the dual variety of $\mathcal{I}mm_q^n$ and give a description of the singular locus of $\mathcal{I}mm_q^n$ and of its $(n-2)$-nd Jacobian locus, namely the zero-locus of the partial derivatives of IMM_q^n of order $(n-2)$.

Before we describe our goals in detail, we briefly present a possible general strategy toward the separation of complexity classes (see e.g. [17] and [18] for details). The main idea is to exploit *pathologies* affecting a sequence of polynomials that is complete for a fixed complexity class, in order to prove that some given sequence of polynomials, not sharing such pathology, does not belong to the complexity class. More precisely, if $g_n \in S^{d_n}\mathbb{C}^{N_n}$ is a sequence of polynomials that is complete for a complexity class $\mathcal{C}$, then a sequence of polynomials $f_m \in S^{e_m}\mathbb{C}^{M_m}$ is in $\mathcal{C}$ if and only if it can be polynomially reduced to g_n, namely if and only if there is a polynomial function $n(m)$ such that, for every m

$$z^{d_{n(m)}-e_m} f_m \in \operatorname{End}(\mathbb{C}^{N_{n(m)}}) \cdot g_{n(m)} \subseteq S^{d_{n(m)}}(\mathbb{C}^{N_{n(m)}}),$$

where z is a *padding variable* and $\mathbb{C}^{M_m} \oplus \mathbb{C}z$ is viewed as a subspace of $\mathbb{C}^{N_{n(m)}}$. When we say *pathology*, we mean a geometric property that is shared by all polynomials in the set $\operatorname{End}(\mathbb{C}^{N_n}) \cdot g_n$ (and possibly other polynomials) but it is not shared by the padded polynomials $z^{d_n - e_m} f_m$, whenever n grows at most polynomially in m; determining such property would prove that the sequence $\{f_m\}$ does not belong to the complexity class $\mathcal{C}$.

The Geometric Complexity Theory (GCT) program (see [20]) focuses on the study of polynomials that are *characterized* by their *symmetry group*, that is the stabilizer under the action of the general linear group of the space generated by the variables. If $f \in S^d W$ and $G_f \subseteq GL(W)$ is its symmetry group, then we say that f is characterized by G_f if it is the only polynomial, up to scale, whose stabilizer contains G_f. The algebraic Peter–Weyl Theorem (see e.g. [21, Ch. 6, Sec. 2.6]) leads to a description of the ring of regular functions on the group orbit $GL(W) \cdot f \subseteq S^d W$ in terms of G_f-invariants; if f is characterized by its stabilizer, then the coordinate ring of the orbit of f is unique as $GL(W)$-module among the coordinate rings of $GL(W)$-orbits in $S^d W$. In particular, sequences of polynomials that are complete for some complexity class and that are char-