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On simple symplectic alternating algebras and their groups of automorphisms

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Let N be any perfect symplectic alternating algebra. We show that N can be embedded into a larger simple alternating algebra S of dimension $7 \cdot (\dim N) + 6$ such that $\mathrm{Aut}\,(S) = \{\mathrm{id}\}$. This answers a question raised in [9]. Building on this result we show moreover that for any finite group G and characteristic c there exists a symplectic alternating algebra L over a field $\mathbb{F}$ of characteristic c such that $\mathrm{Aut}\,(L) = G$.

1. Introduction

A symplectic alternating algebra (SAA) is a symplectic vector space L, whose associated alternating form is nondegenerate, that is furthermore equipped with a binary alternating product $\cdot : L \times L \to L$ with the extra requirement that

$$(x \cdot y, z) = (y \cdot z, x)$$

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for all $x, y, z \in L$. This condition can be expressed equivalently by saying that $(u \cdot x, v) = (u, v \cdot x)$ for all $u, v, x \in L$ or in other words that multiplication from the right is self-adjoint with respect to the alternating form.

Symplectic alternating algebras originate from a study of powerful 2-Engel groups [4,8] and there is a 1-1 correspondence between a certain rich class of powerful 2-Engel 3-groups of exponent 27 and SAAs over the field GF(3).

Let $2n$ be a given even integer and $\mathbb{F}$ a fixed field. Let V be the symplectic vector space over the field $\mathbb{F}$ with a nondegenerate alternating form. Fix some basis $u_1, u_2, \ldots, u_{2n}$ for V. An alternating product $\cdot$ that turns V into a symplectic alternating algebra is uniquely determined by the values

$$\mathcal{P}: \quad (u_i \cdot u_j, u_k), \quad 1 \leq i < j < k \leq 2n.$$

Let L be the resulting symplectic alternating algebra. We refer to the data above as a presentation for L with respect to the basis $u_1, \ldots, u_{2n}$.

Consider the symplectic group $\mathrm{Sp}(V)$. The map $V^3 \to \mathbb{F}$, $(u, v, w) \mapsto (u \cdot v, w)$ is an alternating ternary form and a moment's reflection should convince the reader that there is a 1-1 correspondence between symplectic alternating algebras of dimension $2n$ over $\mathbb{F}$ and orbits in $(\wedge^3 V)^*$ under the natural action of $\mathrm{Sp}(V)$. In particular a symplectic alternating algebra L has a trivial automorphism group if and only if the corresponding orbit in $(\wedge^3 V)^*$ is regular. From this it is not difficult to determine the growth of symplectic alternating algebras. If $m(n)$ is the number of symplectic alternating algebras over a finite field $\mathbb{F}$ then $m(n) = |\mathbb{F}|^{\frac{4n^3}{3} + O(n^2)}$ [7]. Because of the sheer growth, a general classification does not seem to be within reach although this has been done for small values of n. Thus it is not difficult to see that $m(0) = m(1) = 1$ and $m(2) = 2$. For higher dimensions the classification is already difficult. It is though known that when $\mathbb{F} = \mathrm{GF}(3)$ we have $m(3) = 31$ [9]. Some general structure theory is developed in [9] and [10]. In particular there is a dichotomy result that is an analog to a corresponding theorem for Lie algebras, namely that L either contains an abelian ideal or is a direct sum of simple symplectic alternating algebras. We also have that any symplectic algebra that is abelian-by-nilpotent must be nilpotent while this is not the case in general for solvable algebras. We should also mention here that the study of orbits in $\wedge^3 V$ is a classical problem that has been considered by a number of people (see for example [2,5,6]).

For nilpotent symplectic alternating algebras there is a particularly rich structure theory with a number of beautiful results [7]. We can pick a basis $x_1, y_1, \ldots, x_n, y_n$ with the property that $(x_i, x_j) = (y_i, y_j) = 0$ and $(x_i, y_j) = \delta_{ij}$ for $1 \leq i \leq j \leq n$. We refer to a basis of this type as a standard basis. It turns out that for any nilpotent symplectic alternating algebra one can always choose a suitable standard basis such that the chain of subspaces

$$0 = I_0 < I_1 < \ldots < I_n < I_{n-1}^{\perp} < \cdots < I_0^{\perp} = L,$$