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Nonlinear identities with skew derivations

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Let R be a prime ring with the extended centroid $\mathbf{c}$. Suppose that R is acted by a pointed coalgebra with group-like elements acting as automorphisms of R. A generalized polynomial with variables acted by the coalgebra is called an identity if it vanishes on R. We prove the following:

(1) If $\mathbf{c}$ is a perfect field, then any such identity is a consequence of simple basic identities defined in [6] and GPIs of R with variables acted by Frobenius automorphisms.

(2) If $\mathbf{c}$ is not a perfect field, then any such identity is a consequence of simple basic identities defined in [6] and GPIs of R.

With this, we extend Yanai's result [25] to "nonlinear identities". These are actually special instances of our Theorems 1 and 2 below respectively, which extend Kharchenko's theory of differential identities [14,15] to the context of expansion closed word sets introduced in [6].

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0. Results

Throughout, R is a prime ring in the sense that $aRb = 0$ implies $a = 0$ or $b = 0$ for $a, b \in R$. By a (g, h)-derivation of R, where g, h are automorphisms of R, we mean a map $\delta : x \in R \mapsto x^\delta \in U$ such that

$$(x + y)^\delta = x^\delta + y^\delta \quad \text{and} \quad (xy)^\delta = x^g y^\delta + x^\delta y^h \quad \text{for } x, y \in R.$$

A (g, h)-derivation is generally called a *skew* derivation and the ordered pair (g, h) is called the *type* of the skew derivation. Consider a polynomial ϕ with its coefficients in R and with its variables acted by composition products of finitely many automorphisms and skew derivations. If ϕ vanishes identically for any evaluations of its variables in R, then we call ϕ an identity with skew derivations of R. Our aim here is to investigate such identities along the line of Kharchenko's theory of differential identities with automorphisms, which is developed in a series of papers [13–15]. As is shown in this celebrated work, we cannot just sit in R but have to work in the Martindale quotient ring of R. For more generality, let U denote the complete ring of left quotients of R, also called the left Utumi quotient ring of R. We refer to [1,11,19] or [23] for the definition of U. The center of U, denoted by $\mathbf{c}$, is called the extended centroid of R.

Given a set G of automorphisms of R, let L_G denote the set of all (g, h)-derivations with $g, h \in G$. Consider the composition product

$$\Delta := \delta_1 \cdots \delta_n, \qquad \text{where each } \delta_i \in G \cup L_G. \tag{0}$$

Let (g_i, h_i) be the type of $\delta_i \in L_G$. Given a subset S of $\{i : \delta_i \in L_G\}$, let w_S denote the product obtained from Δ by replacing δ_i by g_i for $i \in \{i : \delta_i \in L_G\} \setminus S$. Also let w'_S be the product obtained from Δ by replacing δ_i by h_i for $i \in S$. For $x, y \in R$, we have $(xy)^\Delta = \sum_S x^{w_S} y^{w'_S}$, where S ranges over all subsets of $\{i : \delta_i \in L_G\}$. Let $|S|$ denote the cardinality of S. Then w_S and w'_S contain respectively $|S|$ factors and $(\nu - |S|)$ factors of $\delta_i \in L_G$. So we call w_S and w'_S complementary expansion subwords of Δ. Let g, h be the products obtained from Δ respectively by replacing all $\delta_i \in L_G$ by g_i and by replacing all $\delta_i \in L_G$ by h_i. If $S = \{i : \delta_i \in L_G\}$ then $w_S = \Delta$ and $w'_S = h$. If $S = \emptyset$ then $w'_S = \Delta$ and $w_S = g$. Singling out the two terms with $S = \emptyset$ and $S = \{i : \delta_i \in L_G\}$, we rewrite

$$(xy)^\Delta = x^g y^\Delta + x^\Delta y^h + {\sum_S}' x^{w_S} y^{w'_S} \qquad \text{for } x, y \in R, \tag{0$'$}$$

where $\sum'_S$ ranges all S with $\emptyset \subsetneq S \subsetneq \{i : \delta_i \in L_G\}$. Motivated by this, we consider the following more general notion in order to bring in elements of R or, more generally, of U as coefficients.

Definition 1. ([6], p. 294) By an expansion closed word set, we mean a set Ω of symbols satisfying the following three conditions: