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Infinitely small orbits in two-step nilpotent Lie algebras

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ABSTRACT

In this paper, we consider the open question: is the cortex of the dual of a nilpotent Lie algebra an algebraic set? We give a partial answer by considering the class of two-step nilpotent Lie algebra $\mathfrak{g}$. For this class of Lie algebras we give an explicit algorithm for finding its corresponding cortex. By the way, we prove that either the cortex coincides with the zero set of invariant homogeneous polynomials and in this case is $\mathfrak{z}^{\perp}$ where $\mathfrak{z}$ denotes the center of $\mathfrak{g}$, or it is a proper projective algebraic subset of $\mathfrak{z}^{\perp}$. Finally we materialize our algorithm on examples.

1. Overture

1.1. State of the art

If G is a locally compact group, we define the cortex of G (see [12]) as:

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$$cor(G) = \{\pi \in \widehat{G}, \text{ which cannot be Hausdorff-separated from the identity representation } \mathbf{1}_G\},$$

where $\widehat{G}$ is the dual of G (set of class of unitary irreducible representations of G), that is, $\pi \in cor(G)$ if and only if for all neighborhood V of $\mathbf{1}_G$ and for each neighborhood U of π, one has $V \cap U$ is non-empty set. Note that $\widehat{G}$ is equipped with the topology of Fell which can be described in terms of weak containment (see [9]). This topology is in general not separated, however if G is abelian, $\widehat{G}$ is separated and $cor(G) = \{\mathbf{1}_G\}$. When G is a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, the Kirillov theory says that $\mathfrak{g}^*/Ad^*(G)$ and $\widehat{G}$ are homomorphic, where $Ad^*(G)$ denotes the coadjoint representation of G on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. Hence, for this class of Lie groups, $cor(G)$ can be identified with certain $Ad^*(G)$-invariant subset of $\mathfrak{g}^*$. From [4], one introduces the cortex of $\mathfrak{g}^*$ as

$$Cor(\mathfrak{g}^*) = \{\ell = \lim_{m\to\infty} Ad^*_{s_m}(\ell_m), \text{ where } \{s_m\} \subset G \text{ and } \{\ell_m\} \subset \mathfrak{g}^* \text{ such that } \lim_{m\to\infty} \ell_m = 0\}$$

and we have $\pi_\ell \in cor(G)$ if and only if $\ell \in Cor(\mathfrak{g}^*)$. Hence one can investigate the parametrization of $\mathfrak{g}^*$ to determine such a set as in [5], in which the authors define the cortex $C_V(G)$ of a representation of a locally compact group G on a finite-dimensional vector space V as the set of all $v \in V$ for which $G.v$ and $\{0\}$ cannot be Hausdorff-separated in the orbit-space V/G. They give a precise description of $C_V(G)$ in the case $G = \mathbb{R}$. Moreover, they consider the subset $IC_V(G)$ of V consisting of the common zeroes of all G-invariant polynomials p on V with $p(0) = 0$. They show that $IC_V(G) = C_V(G)$ when G is a nilpotent Lie group of the form $G = \mathbb{R} \ltimes \mathbb{R}^n$ and $V = \mathfrak{g}^*$ the dual of the Lie algebra $\mathfrak{g}$. This fails for a general nilpotent Lie group, even in the case of two-step nilpotent Lie group. In [4], the authors give a counter-example of 8-dimensional Lie algebra $\mathfrak{g}$ for which $Cor(\mathfrak{g}^*) \subsetneq ICor(\mathfrak{g}^*)$ and they show that the cortex of any two-step nilpotent Lie algebra $\mathfrak{g}$ is the closure of the set:

$$\{ad^*_X(\ell),\, X \in \mathfrak{g},\, \ell \in \mathfrak{g}^*\} \qquad (1).$$

In [10], the authors show that the cortex of a connected and simply connected nilpotent Lie group is a semi-algebraic set. In [8], one gives a family $(\mathfrak{g}_d)_{d\geq 2}$ of $8d$-dimensional two-step nilpotent Lie algebras, which constitute a generalization of the example cited in [4], and such that for each d, one has $Cor(\mathfrak{g}^*_d) \subsetneq ICor(\mathfrak{g}^*_d)$ and $Cor(\mathfrak{g}^*_d)$ is a projective algebraic subset in $\mathfrak{g}^*$. In [7] one gives an explicit description of the cortex of certain class of exponential Lie algebras $\mathfrak{g}$ by investigating the tools of parametrization of the coadjoint orbits in $\mathfrak{g}^*$.