Journal of Algebra 457 (2016) 7–17

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On Birkhoff's quasigroup axioms

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A R T I C L E I N F O

Article history:
Received 13 November 2015
Available online 21 March 2016
Communicated by Alberto Elduque

MSC:
20N05

Keywords:
(Left) quasigroup
(Equational) quasigroup
Division groupoid
Cancellation groupoid

A B S T R A C T

Birkhoff defined a *quasigroup* as an algebra $(Q, \cdot, \backslash, /)$ that satisfies the following six identities: $x \cdot (x\backslash y) = y$, $(y/x) \cdot x = y$, $x\backslash(x \cdot y) = y$, $(y \cdot x)/x = y$, $x/(y\backslash x) = y$, and $(x/y)\backslash x = y$. We investigate triples and tetrads of identities composed of these six, emphasizing those that axiomatize the variety of quasigroups.

1. Introduction and terminology

A *binary groupoid*, (G, A), is a non-empty set G, together with a binary operation A. It is customary to omit the adjective "binary" and to refer to these simply as "groupoids". We will include the adjective "binary" in those instances where we wish to emphasize it.

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http://dx.doi.org/10.1016/j.jalgebra.2016.02.024

Moufang [9,18] defined a *quasigroup* as a groupoid $(Q, \circ)$ in which, for all $a, b \in Q$, there exist unique solutions $x, y \in Q$ to the equations $x \circ a = b$ and $a \circ y = b$. Moufang also included two of the laws that eventually came to be called "the Moufang laws" in her definition of quasigroup, viz. $(x \cdot y) \cdot (z \cdot x) = x \cdot ((y \cdot z) \cdot x)$. The definition of "quasigroup" has evolved a bit over the years, so that today the word "quasigroup" refers to objects slightly more structured than Moufang's, as we shall see, but that don't necessarily satisfy the Moufang laws. Fuller accounts of basic terms and concepts, as well as more comprehensive historical overviews, can be found in [2,5,13,14,20].

Bates and Kiokemeister [1] noticed that the class of quasigroups defined qua Moufang is not closed with respect to the taking of homomorphic images. Explicitly, homomorphic images of quasigroups (defined in this manner) are only division groupoids (definition given below).

This algebraic infelicity was fixed by Birkhoff who defined a quasigroup using three binary operations and the six identities given above [6,7]. Shortly thereafter, T. Evans [11] shortened Birkhoff's list of six identities to four (we give them, below). It follows, then, from the classic universal algebra theorem named for Birkhoff, that the class of (equational) quasigroups is a variety and is thus closed with respect to the taking of homomorphic images [7,8]. In this paper we give two triples of identities from Birkhoff's list of six, as well as nine tetrads, each of which axiomatizes the variety of quasigroups.

Let $(Q, \cdot)$ be a groupoid. We use the standard notation $L_a x = a \cdot x$ for all $x \in Q$, to denote left translation by the fixed element $a \in Q$. Right translations are defined analogously. We define middle translations, P_a, as follows: $x \cdot P_a x = a$ for all $x \in Q$ [3]. The definition of middle ternary relation for groupoids (an analogue of middle quasigroup translation) is given in [22].

Finally, we recall another definition of "quasigroup" which is equivalent to the equational definition given above: a binary groupoid (Q, A) with binary operation A is called a *binary quasigroup* if, in the equality $A(x_1, x_2) = x_3$, knowledge of any two of x_1, x_2, and x_3 specifies the third uniquely [4].

From this definition it follows that with a given binary quasigroup (Q, A), it is possible to associate $(3! - 1)$ others, the so-called *parastrophes of* (Q, A)—there are six of them, as we shall see:

1. $A(x_1, x_2) = x_3 \Longleftrightarrow$
2. $A^{(12)}(x_2, x_1) = x_3 \Longleftrightarrow$
3. $A^{(13)}(x_3, x_2) = x_1 \Longleftrightarrow$
4. $A^{(23)}(x_1, x_3) = x_2 \Longleftrightarrow$
5. $A^{(123)}(x_2, x_3) = x_1 \Longleftrightarrow$
6. $A^{(132)}(x_3, x_1) = x_2$

[28, p. 230], [2, p. 18].