Journal of Algebra 457 (2016) 18–25

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Derivations of the moduli algebras of weighted homogeneous hypersurface singularities

Stephen S.-T. Yau [a,*,1], Huaiqing Zuo [b,*,2]

[a] Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, PR China
[b] Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, PR China

ARTICLE INFO

Article history:
Received 29 December 2014
Available online 21 March 2016
Communicated by Steven Dale Cutkosky

Keywords:
Derivations
Moduli algebra
Isolated singularity

ABSTRACT

Let $R = \mathbb{C}[x_1, x_2, \cdots, z_n]/(f)$ where f is a weighted homogeneous polynomial defining an isolated singularity at the origin. Then R, and $Der(R)$, the Lie algebra of derivations on R, are graded. It is well-known that $Der(R)$ has no negatively graded component [10]. J. Wahl conjectured that the above fact is still true in higher codimensional case provided that $R = \mathbb{C}[x_1, x_2, \cdots, x_n]/(f_1, f_2, \cdots, f_m)$ is an isolated, normal and complete intersection singularity and $f_1, f_2, \cdots, f_m$ are weighted homogeneous polynomials with the same weight type $(w_1, w_2, \cdots, w_n)$. On the other hand the first author Yau conjectured that the moduli algebra $A(V) = \mathbb{C}[x_1, x_2, \cdots, x_n]/(\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ has no negatively weighted derivations where f is a weighted homogeneous polynomial defining an isolated singularity at the origin. Assuming this conjecture has a positive answer, he gave a characterization of weighted homogeneous hypersurface singularities only using the Lie algebra $Der(A(V))$ of derivations on $A(V)$. The conjecture of Yau can be thought as an Artinian analogue of J. Wahl's conjecture. For the low embedding dimension, the Yau conjecture has a positive answer. In this paper we prove this conjecture for any high-dimensional sin-

* Corresponding authors.
E-mail addresses: yau@uic.edu (S.S.-T. Yau), hqzuo@math.tsinghua.edu.cn (H. Zuo).

[1] This work is supported by NSFC (grant no. 11531007) and the start-up fund from Tsinghua University.
[2] This work is supported by the NSFC (grant nos. 11401335, 11531007), Tsinghua University Initiative Scientific Research Program and the start-up fund from Tsinghua University.

http://dx.doi.org/10.1016/j.jalgebra.2016.03.003

gularities under the condition that the lowest weight is bigger than or equal to half of the highest weight.

1. Introduction

Let A be a weighted zero dimensional complete intersection, i.e., a commutative algebra of the form

$$A = \mathbb{C}[x_1, x_2, \cdots, x_n]/I$$

where the ideal I is generated by a regular sequence of length n, $(f_1, f_2, \cdots, f_n)$. Here the variables have strictly positive integral weights, denoted by $wt(x_i) = w_i$, $1 \leq i \leq n$, and the equations are weighted homogeneous with respect to these weights. They are arranged for future convenience in the decreasing order of the degrees: $p_i := deg f_i$, $i = 1, 2, \cdots, n$ and $p_1 \geq p_2 \geq \cdots \geq p_n$. Consequently the algebra A is graded and one may speak about its homogeneous degree k derivations (k is an integer). A linear map $D : A \to A$ is a derivation if $D(ab) = D(a)b + aD(b)$, for any $a, b \in A$. D belongs to $Der^k(A)$ if $D : A^* \to A^{*+k}$. One of the most important open problems in rational homotopy theory is related to the vanishing of the above derivations in strictly negative degrees:

Halperin Conjecture. *(See [5].) If A is as above, then $Der^{<0}(A) = 0$.*

Assuming that all the weights w_i are even, this has the following topological interpretation. If a space X has $H^*(X, \mathbb{C}) = A$ as graded algebras, then it is known that the vanishing of $Der^{<0}(A) = 0$ implies the collapsing at the E_2-term of the Serre spectral sequence with $\mathbb{C}$-coefficients of any orientable fibration having X as fiber. Actually the above collapsing properties also implies vanishing properties when $\mathbb{C}$ is replaced by $\mathbb{Q}$ and X a rational space, see e.g. [5]. The Halperin Conjecture has been verified in several particular cases [10]:

1) equal weights ($w_1 = w_2 = \cdots = w_n$), see [14];
2) $n = 2, 3$, see [9,3];
3) "fibered" algebras see [4];
4) assuming $\mathbb{C}[x_1, x_2, \cdots, x_n]/(f_1, f_2, \cdots, f_{n-1})$ is reduced, see [6].
5) homogeneous spaces of equal rank compact connected Lie groups ($A = H^*(G/K)$), see [8].

On the other hand S.S.-T. Yau discovered independently the following conjecture on the nonexistence of the negative weight derivation from his work on *Lie* algebras of