Journal of Algebra 457 (2016) 45–72

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On Clifford theory with Galois action

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A R T I C L E I N F O

Article history:
Received 10 July 2015
Available online 22 March 2016
Communicated by Michel Broué

MSC:
20C15

Keywords:
Brauer–Clifford group
Clifford theory
Character theory of finite groups
Galois theory
Schur indices

A B S T R A C T

Let $\widehat{G}$ be a finite group, N a normal subgroup of $\widehat{G}$ and $\vartheta \in \operatorname{Irr} N$. Let $\mathbb{F}$ be a subfield of the complex numbers and assume that the Galois orbit of ϑ over $\mathbb{F}$ is invariant in $\widehat{G}$. We show that there is another triple $(\widehat{G}_1, N_1, \vartheta_1)$ of the same form, such that the character theories of $\widehat{G}$ over ϑ and of $\widehat{G}_1$ over ϑ_1 are essentially "the same" *over the field* $\mathbb{F}$ and such that the following holds: $\widehat{G}_1$ has a cyclic normal subgroup C contained in N_1, such that $\vartheta_1 = \lambda^{N_1}$ for some linear character λ of C, and such that N_1/C is isomorphic to the (abelian) Galois group of the field extension $\mathbb{F}(\lambda)/\mathbb{F}(\vartheta_1)$. More precisely, having "the same" character theory means that both triples yield the same element of the Brauer–Clifford group $\operatorname{BrCliff}(G, \mathbb{F}(\vartheta))$ defined by A. Turull.

1. Introduction

1.1. Motivation

Clifford theory is concerned with the characters of a finite group lying over one fixed character of a normal subgroup. So let $\widehat{G}$ be a finite group and $N \trianglelefteq \widehat{G}$ a normal subgroup.

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[1] Partially supported by the DFG (Project: SCHU 1503/6-1).

http://dx.doi.org/10.1016/j.jalgebra.2016.03.008

Let $\vartheta \in \operatorname{Irr} N$, where $\operatorname{Irr} N$ denotes the set of irreducible complex-valued characters of the group N, as usual. We write $\operatorname{Irr}(\widehat{G} \mid \vartheta)$ for the set of irreducible characters of $\widehat{G}$ which lie above ϑ in the sense that their restriction to N has ϑ as constituent.

In studying $\operatorname{Irr}(\widehat{G} \mid \vartheta)$, it is usually no loss of generality to assume that ϑ is invariant in $\widehat{G}$, using the well known *Clifford correspondence* [9, Theorem 6.11]. In this situation, $(\widehat{G}, N, \vartheta)$ is often called a character triple. Then a well known theorem tells us that there is an "isomorphic" character triple $(\widehat{G}_1, N_1, \vartheta_1)$ such that $N_1 \subseteq \mathbf{Z}(\widehat{G}_1)$ [9, Theorem 11.28]. Questions about $\operatorname{Irr}(\widehat{G} \mid \vartheta)$ can often be reduced to questions about $\operatorname{Irr}(\widehat{G}_1 \mid \vartheta_1)$, which are usually easier to handle. This result is extremely useful, for example in reducing questions about characters of finite groups to questions about characters of finite simple or quasisimple groups.

Some of these questions involve Galois automorphisms or even Schur indices [13,21] (over some fixed field $\mathbb{F} \subseteq \mathbb{C}$, say). Unfortunately, both of the above reductions are not well behaved with respect to Galois action on characters and other rationality questions (like Schur indices of the involved characters). The first reduction (Clifford correspondence) can be replaced by a reduction to the case where ϑ is semi-invariant over the given field $\mathbb{F}$ [15, Theorem 1]. (This means that the Galois orbit of ϑ is invariant in the group $\widehat{G}$.)

Now assuming that the character triple $(\widehat{G}, N, \vartheta)$ is such that ϑ is semi-invariant in $\widehat{G}$ over some field $\mathbb{F}$, usually we cannot find a character triple $(\widehat{G}_1, N_1, \vartheta_1)$ with $N_1 \subseteq \mathbf{Z}(\widehat{G}_1)$, and such that these character triples are "isomorphic over the field $\mathbb{F}$". We will give an exact definition of "isomorphic over $\mathbb{F}$" below, using machinery developed by Alexandre Turull [23,24]. For the moment, it suffices to say that a correct definition should imply that $\widehat{G}/N \cong \widehat{G}_1/N_1$ and that there is a bijection between $\bigcup_\alpha \operatorname{Irr}(\widehat{G} \mid \vartheta^\alpha)$ and $\bigcup_\alpha \operatorname{Irr}(\widehat{G}_1 \mid \vartheta_1^\alpha)$ (unions over a Galois group) commuting with field automorphisms over $\mathbb{F}$ and preserving Schur indices over $\mathbb{F}$. Now if, for example, $\mathbb{Q}(\vartheta) = \mathbb{Q}(\sqrt{5}\,)$ (say), then it is clear that we cannot have $\mathbb{Q}(\vartheta) = \mathbb{Q}(\vartheta_1)$ with ϑ_1 linear. The main result of this paper, as described in the abstract, provides a possible substitute: At least we can find an "isomorphic" character triple $(\widehat{G}_1, N_1, \vartheta_1)$, where N_1 is cyclic by abelian and ϑ_1 is induced from a cyclic normal subgroup of $\widehat{G}_1$. This result is probably the best one can hope for, if one wants to take into account Galois action and Schur indices.

1.2. Notation

To state the main result precisely, we need some notation. Instead of character triples, we find it more convenient to use Clifford pairs as introduced in [12]. We recall the definition. Let $\widehat{G}$ and G be finite groups and let $\kappa\colon \widehat{G} \to G$ be a surjective group homomorphism with kernel $\operatorname{Ker} \kappa = N$. Thus

$$1 \longrightarrow N \longrightarrow \widehat{G} \xrightarrow{\kappa} G \longrightarrow 1$$

is an exact sequence, and $\widehat{G}/N \cong G$ via κ. We say that (ϑ, κ) is a **Clifford pair** over G. (Note that $\widehat{G}$, G and N are determined by κ as the domain, the image and the kernel