Journal of Algebra 457 (2016) 73–105

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A new family of modules over the Virasoro algebra

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ARTICLE INFO

Article history:
Received 27 June 2015
Available online 22 March 2016
Communicated by Volodymyr Mazorchuk

MSC:
17B10
17B20
17B65
17B66
17B68

Keywords:
Virasoro algebra
Simple modules
Maximal submodules

ABSTRACT

In this paper, we constructed a new family of Virasoro modules $\Omega(\lambda, \alpha, h)$ parameterized by complex numbers $\lambda \neq 0$, $\alpha \in \mathbb{C}$ and a polynomial $h(t)$. They are a special type of modules on which the non-central Cartan part acts freely of infinite rank. These modules are natural generalizations of the modules $\Omega(\lambda, \alpha)$ defined and studied in [5], [11] and [19]. We determine the sufficient and necessary conditions for them to be irreducible. When they are reducible, we also determine all their maximal submodules and corresponding simple quotients explicitly. For some special cases, we can determine all submodules.

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http://dx.doi.org/10.1016/j.jalgebra.2016.02.020
0021-8693/

1. Introduction

Let $\mathbb{C}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ be the sets of all complex numbers, all integers, all non-negative integers and all positive integers, respectively. The **Virasoro algebra** Vir is an infinite dimensional Lie algebra over the complex numbers $\mathbb{C}$, with basis $\{L_n, C \mid n \in \mathbb{Z}\}$ and defining relations

$$[L_m, L_n] = (n-m)L_{n+m} + \delta_{n,-m}\frac{m^3-m}{12}C, \quad m, n \in \mathbb{Z},$$
$$[C, L_m] = 0, \quad m \in \mathbb{Z},$$

which is the universal central extension of the infinite dimensional **Witt algebra** of rank 1.

The theory of weight Virasoro modules with finite-dimensional weight spaces is fairly well developed (see [8] and references therein). In 1992, O. Mathieu [13] classified all simple Harish–Chandra modules, that is, simple modules with finite-dimensional weight spaces, over the Virasoro algebra, which was conjectured by Kac [7]. Recently, many authors constructed many classes of simple non-Harish–Chandra modules, including simple weight modules with infinite-dimensional weight spaces (see [3,4,9,12,23]) and simple non-weight modules (see [1,10,9,11,14,15,18–22]).

Recently, some authors constructed and studied simple modules on which the Cartan subalgebras act freely, for the simple finite dimensional Lie algebras ([16,17]), the Witt algebras ([19]), the Heisenberg–Virasoro algebra and the W-algebra $W(2,2)$ ([2]). In particular, one obtains new simple modules for the algebra $W(2,2)$. Since the Virasoro algebra is a subalgebra of $W(2,2)$, the $W(2,2)$-modules can be viewed as Virasoro modules. We denote these Vir-modules by $\Omega(\lambda, \alpha, h)$, where $\lambda, \alpha \in \mathbb{C}$ with $\lambda \neq 0$ and h is a polynomial in $\mathbb{C}[t]$. The modules $\Omega(\lambda, \alpha, h)$ are similar with those modules $\Omega(\lambda, \alpha)$ constructed and studied in [5,11,19]. In this paper, we will investigate the simplicity and structure for the modules $\Omega(\lambda, \alpha, h)$.

In Section 2, we recall some known results on the module $\Omega(\lambda, \alpha)$ and give the definition of the modules $\Omega(\lambda, \alpha, h)$. The simplicity are determined in Section 3. More precisely, it was shown that the Vir-module $\Omega(\lambda, \alpha, h)$ is simple if and only if $\alpha \neq 0$ and $\deg(h) = 1$. Then we determine the maximal submodules for all reducible ones in Section 4. As examples, we consider some special modules and give explicit descriptions of all their submodules in the last section.

2. The Vir-modules $\Omega(\lambda, \alpha, h)$

Since our modules $\Omega(\lambda, \alpha, h)$ are similar and closely related to the modules $\Omega(\lambda, \alpha)$, we first recall some known results about these modules.