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# On the splitting fields of generic elements in Zariski dense subgroups

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ABSTRACT

Let $G$ be a connected, absolutely almost simple, algebraic group defined over a finitely generated, infinite field $K$, and let $\Gamma$ be a Zariski dense subgroup of $G(K)$. We show, apart from some few exceptions, that the commensurability class of the field $\mathcal{F}$ given by the compositum of the splitting fields of characteristic polynomials of generic elements of $\Gamma$ determines the group $G$ up to isogeny over the algebraic closure of $K$.


## 1. Introduction

The inverse spectral theory problem in Riemannian geometry is to recover properties of a Riemannian manifold from the knowledge of the spectra of natural differential operators associated to the manifold. This problem has been intensely studied in the context of Riemannian locally symmetric spaces by various authors.

In [4] G. Prasad and A.S. Rapinchuk address this question in full generality. They introduce a notion of weak commensurability for Zariski dense subgroups in absolutely

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almost simple connected algebraic groups. This notion is weaker than commensurability, but they showed that weak commensurability of arithmetic lattices implies commensurability in many instances.

Assuming the validity of Schanuel's conjecture on transcendental numbers in the case of higher rank lattices, they first show that if the locally symmetric spaces defined by arithmetic lattices are isospectral with respect to the Laplacian associated to the invariant metric acting on the space of smooth functions, then the lattices are weakly commensurable. From this, they obtain commensurability type results for these lattices.

In [1], the authors considered representation equivalent lattices. If two uniform lattices are representation equivalent, then the corresponding Riemannian locally symmetric spaces are 'strongly isospectral'; in particular, they are isospectral for the Laplacian.

One of the major advantages for considering the stronger but natural notion of representation equivalence is that commensurability type results for representation equivalent lattices can be obtained without appealing to Schanuel's conjecture. It can be seen by an application of the trace formula, that representation equivalent uniform lattices are characteristic equivalent.

The notion of characteristic equivalence of lattices introduced in [1] is as follows: for an algebraic group $G$ defined over a field $K$ and an element $\gamma \in G(K)$, let $P(\gamma, Ad)$ denote the characteristic polynomial of $\gamma$ in $G(K)$ with respect to the adjoint representation $Ad$ of $G$ on its Lie algebra. Let $G_1,\ G_2$ be connected absolutely almost simple algebraic groups defined over a number field $K$. Two Zariski dense subgroups $\Gamma_i \subset G_i(K)$, $i = 1, 2$ are said to be *characteristic equivalent* if the collection of characteristic polynomials $P(\gamma, Ad)$ of elements $\gamma$ in $\Gamma_1$ (resp. $\Gamma_2$) are equal.

The concept of characteristic equivalence is stronger than that of weak commensurability. From characteristic equivalence, the commensurability results of [4] follow more directly and easily using the methods of [4].

For an arithmetic lattice, the lengths of closed geodesics can be expressed as a sum of logarithms of algebraic numbers, which arise as the evaluation of the roots on lattice elements. In a subsequent paper [5], Prasad and Rapinchuk studied the compositum of the fields generated over the field of algebraic numbers, by the lengths of closed geodesics on the locally Riemannian symmetric space defined by the lattice. Upon certain hypothesis on the Weyl group and conditional on the validity of Schanuel's conjecture on transcendental numbers, they showed that the fields are quite different, provided the lattices are not commensurable.

In this paper, we examine an analogous question, in the context of characteristic equivalence. We consider the compositum of the splitting fields of the characteristic polynomials of generic elements of $G(K)$ contained in the Zariski dense subgroup $\Gamma \subset G(K)$. The roots of the characteristic polynomial of an element $\gamma$ with respect to the adjoint representation are the evaluations of the roots (with respect to the torus generated by $\gamma$) evaluated at $\gamma$. Apart from some few exceptions, we show that this field determines the 'commensurability class' of $\Gamma$, i.e., the group $G$ up to isogeny over the algebraic closure