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Polynomial recognition of cluster algebras of finite type

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A B S T R A C T

Cluster algebras are a recent topic of study and have been shown to be a useful tool to characterize structures in several knowledge fields. An important problem is to establish whether or not a given cluster algebra is of finite type. Using the standard definition, the problem is infeasible since it uses mutations that can lead to an infinite process. Barot, Geiss and Zelevinsky [1] presented an easier way to verify if a given algebra is of finite type, by testing if all chordless cycles of the graph related to the algebra are cyclically oriented and if there exists a positive quasi-Cartan companion of the skew-symmetrizable matrix related to the algebra. We develop an algorithm that verifies these conditions and decides whether or not a cluster algebra is of finite type in polynomial time. The second part of the algorithm is used to prove that the more general problem to decide if a matrix has a positive quasi-Cartan companion is in $\mathcal{NP}$ class.

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1. Introduction

Matrices can be used to represent various structures arising in combinatorics, including graphs and algebras, such as cluster algebras. The latter can be defined using a directed graph $G(B)$, called quiver, and consequently by an adjacency matrix, where rows and columns represent the vertices and the positive values at positions (i, j) represent the quantity of edges between associated vertices of the graph.

In 2002, Sergey Fomin and Andrei Zelevinsky [9] introduced a class of commutative algebras called cluster algebras. These algebras have a strong combinatorial structure. They are tools to study questions of dual canonical bases and positivity of semisimple Lie groups. Cluster algebras are defined recursively via commutative algebras with a distinct set of generating variables (*cluster variables*) grouped into overlapping subsets (*clusters*) of fixed cardinality.

A basic feature of cluster algebra class is that both the generators and the relationships between them are not given from the start, but are produced by an elementary iterative process of seed mutation. This process is somewhat counterintuitive, but it seems to encode a universal phenomenon in some way. This may explain the accelerated development of the cluster algebra theme in areas such as combinatorics, physics, mathematics (especially geometry), among others as discussed in [11,12,14]. This algebra can be defined using a skew-symmetrizable matrix, as we will see in Section 3.

The notion of quasi-Cartan matrices was introduced by Barot, Geiss and Zelevinsky [1]. A quasi-Cartan matrix is a symmetrizable matrix with all entries of the main diagonal equal to 2. The authors show some properties of the matrices from the mathematical point of view. A quasi-Cartan companion is a quasi-Cartan matrix associated with skew-symmetrizable matrix as we will see in Section 2.

One can decide whether a cluster algebra is of finite type (has a finite number of cluster variables) by deciding whether or not the skew-symmetrizable matrix is associated with a cyclically oriented graph and has a positive quasi-Cartan companion. By the Sylvester criterion [4], a symmetric matrix is positive if all its leading principal submatrices have positive determinant.

In this paper, we present three algorithms of polynomial time complexity. The first one can be used to decide whether or not a quasi-Cartan companion matrix is positive. This is used as a certificate to prove that the problem of setting whether a positive quasi-Cartan companion there exists or not belongs to the $\mathcal{NP}$ class of problems. The second algorithm can be used to establish whether or not there exists a positive quasi-Cartan companion of a skew-symmetrizable matrix associated with a cyclically oriented graph. To verify that an oriented graph is cyclically oriented, we will use the algorithm and results presented in [3]. The last algorithm can be used to decide whether or not a cluster algebra is of finite type. This is used to prove that the problem belongs to the $\mathcal{P}$ class of problems. For more information about $\mathcal{P}$, $\mathcal{NP}$ and $\mathcal{NP}$-complete classes, see [6,17].

The remainder of the paper is organized as follows: Section 2 presents the preliminary concepts; Section 3 defines what is a cluster algebra of finite type and presents some