Journal of Algebra 459 (2016) 189–212

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Pure projective modules over chain domains with Krull dimension ☆

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ARTICLE INFO

Article history:
Received 6 August 2015
Available online 28 April 2016
Communicated by J.T. Stafford

MSC:
primary 16D70
16P70

Keywords:
Chain domain
Pure projective module
Uniserial module
Finitely presented module

ABSTRACT

We will prove that over a chain domain with Krull dimension each pure projective module decomposes into a direct sum of finitely presented modules.

1. Introduction

A ring is said to be a chain ring if its right ideals are linearly ordered by inclusion, and the same holds true for its left ideals.

☆ The paper was written during two stays of the second author in ECI, Prague. Both authors were partially supported by GA ČR P201/12/G028.
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http://dx.doi.org/10.1016/j.jalgebra.2016.04.010
0021-8693/

A clever trick by Drozd (see [12, Theorem 2.1]) shows that every matrix over a chain ring R can be diagonalized by elementary row and column transformations. In particular every finitely presented right module over R is isomorphic to a direct sum of cyclically presented modules R/rR, $r \in R$; furthermore (see [5, Theorem 9.19]) this decomposition is essentially unique. It follows from Warfield (see [18, 33.6]) that every pure projective module over a chain ring is isomorphic to a direct summand of a direct sum of uniserial modules R/rR. If each module R/rR has a local endomorphism ring, then the extended version of the Crawley–Jønsson theorem (see [5, Theorem 5.2]) shows that every pure projective module is isomorphic to a direct sum of finitely presented modules. For example this is the case when R is a commutative chain ring.

However for a general chain ring the situation is less orderly. For instance, Puninski [13] constructed a non-finitely generated uniserial pure projective module over a nearly simple chain domain; and also (see [14]) found a pure projective module without an indecomposable decomposition over an exceptional chain ring.

In this paper we will completely characterize chain domains R such that every pure projective R-module is isomorphic to a direct sum of finitely presented modules. Namely this is the case if and only if R contains no idempotent 2-sided ideal RrR, where $0 \neq r \in R$ is not a unit. For instance each chain domain with Krull dimension possesses this property.

To prove something nice we have to avoid complicated things, so the bulk of the paper is spent analyzing various anomalies of direct sum decompositions of serial modules. The main tool in this analysis is the so-called dimension theory for serial modules, recently developed by Facchini and Příhoda [6]. It states (see Proposition 4.1 for an exact statement) that pure projective modules over a chain ring R are classified by tuples of dimensions: one for each module R/rR with a local endomorphism ring, and two for each module R/rR whose endomorphism ring is not local.

Any countably generated pure projective module can be constructed as a colimit of a special directed system (called a Mittag–Leffler system) of morphisms between finitely presented modules. In this paper we will suggest a very abstract version of this construction that covers all known examples of strange pure projective modules over chain rings. By evaluating it for various dimensions we will identify the main obstacles to the existence of a perfect (hence trivial) decomposition theory of pure projectives.

Because these anomalies are of great interest we will demonstrate non-regular behavior of pure projective modules by various examples. As a main running example we will use the chain domain (constructed by Dubrovin – see [3], or better [4]) which is associated with a special embedding of the group ring of the trefoil group into a skew field.

We are indebted to the referee for a very careful reading of the manuscript, which greatly improved the presentation.