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Universal deformations of the finite quotients of the braid group on 3 strands

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ABSTRACT

We prove that the quotients of the group algebra of the braid group on 3 strands by a generic quartic and quintic relation respectively have finite rank. This is a special case of a conjecture by Broué, Malle and Rouquier for the generic Hecke algebra of an arbitrary complex reflection group. Exploring the consequences of this case, we prove that we can determine completely the irreducible representations of this braid group of dimension at most 5, thus recovering a classification of Tuba and Wenzl in a more general framework.

1. Introduction

In 1999 I. Tuba and H. Wenzl classified the irreducible representations of the braid group B_3 of dimension k at most 5 over an algebraically closed field K of any characteristic (see [19]) and, therefore, of $PSL_2(\mathbb{Z})$, since the quotient group B_3 modulo its center is isomorphic to $PSL_2(\mathbb{Z})$. Recalling that B_3 is given by generators s_1 and s_2 that satisfy the relation $s_1s_2s_1 = s_2s_1s_2$, we assume that $s_1 \mapsto A$, $s_2 \mapsto B$ is an irreducible representation of B_3, where A and B are invertible $k \times k$ matrices over K satisfying

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$ABA = BAB$. I. Tuba and H. Wenzl proved that A and B can be chosen to be in *ordered triangular form*[1] with coefficients completely determined by the eigenvalues (for $k \leq 3$) or by the eigenvalues and by the choice of a kth root of $\det A$ (for $k > 3$). Moreover, they proved that such irreducible representations exist if and only if the eigenvalues do not annihilate some polynomials P_k in the eigenvalues and the choice of the kth root of $\det A$, which they determined explicitly.

At this point, a number of questions arise: what is the reason we do not expect their methods to work for any dimension beyond 5 (see [19], Remark 2.11, 3)? Why are the matrices in this neat form? In [19], Remark 2.11, 4 there is an explanation for the nature of the polynomials P_k. However, there is no argument connected with the nature of P_k that explains the reason why these polynomials provide a necessary condition for a representation of this form to be irreducible. In this paper we answer these questions by recovering this classification of the irreducible representations of the braid group B_3 as a consequence of the freeness conjecture for the generic Hecke algebra of the finite quotients of the braid group B_3, defined by the additional relation $s_i^k = 1$, for $i = 1, 2$ and $2 \leq k \leq 5$. For this purpose, we first prove this conjecture for $k = 4, 5$ (the rest of the cases are known by previous work). The fact that there is a connexion between the classification of the irreducible representations of B_3 of dimension at most 5 and its finite quotients has already been suspected by I. Tuba and H. Wenzl (see [19], Remark 2.11, 5).

More precisely, there is a Coxeter's classification of the finite quotients of the braid group B_n on n strands by the additional relation $s_i^k = 1$, for $i = 1, 2$ (see [7]); these quotients are finite if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$. If we exclude the obvious cases $n = 2$ and $k = 2$, which lead to the cyclic groups and to the symmetric groups respectively, there is only a finite number of such groups, which are irreducible complex reflection groups: these are the groups G_4, G_8 and G_{16}, for $n = 3$ and $k = 3, 4, 5$ and the groups G_{25}, G_{32} for $n = 4, 5$ and $k = 3$, as they are known in the Shephard–Todd classification (see [18]). Therefore, if we restrict ourselves to the case of B_3, we have the finite quotients W_k, for $2 \leq k \leq 5$, which are the groups $\mathfrak{S}_3$, G_4, G_8 and G_{16}, respectively.

We set $R_k = \mathbb{Z}[a_{k-1}, ..., a_1, a_0, a_0^{-1}]$, for $k = 2, 3, 4, 5$ and we denote by H_k the *generic Hecke algebra* of W_k; that is the quotient of the group algebra $R_k B_3$ by the relations $s_i^k = a_{k-1}s_i^{k-1} + ... + a_1 s_i + a_0$, for $i = 1, 2$. We assume we have an irreducible representation of B_3 of dimension k at most 5. By the Cayley–Hamilton theorem of linear algebra, the image of a generator under such a representation is annihilated by a monic polynomial $m(X)$ of degree k, therefore this representation has to factorize through the corresponding Hecke algebra H_k. As a result, if $\theta : R_k \to K$ is a specialization of H_k such that $a_i \mapsto m_i$, where m_i are the coefficients of $m(X)$, the irreducible representations of B_3 of dimension k are exactly the irreducible representations of $H_k \otimes_\theta K$ of dimension k. A conjecture of Broué, Malle and Rouquier states that H_k is free as R_k-module of rank

[1] Two $k \times k$ matrices are in ordered triangular form if one of them is an upper triangular matrix with eigenvalue λ_i as ith diagonal entry, and the other is a lower triangular matrix with eigenvalue λ_{k+1-i} as ith diagonal entry.