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A triviality criterion for $\mathbb{A}^2$-fibrations over a ring containing $\mathbb{Q}$

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Let R be an arbitrary commutative ring with unity containing $\mathbb{Q}$ and let A be a stably trivial $\mathbb{A}^2$-fibration over R. In this paper we prove that A is trivial if and only if it has a fixed point free locally nilpotent R-derivation.

1. Introduction

Throughout, all considered rings are commutative with unity. Given a ring R and a positive integer n we denote by $R^{[n]}$ the polynomial R-algebra in n variables. For an R-algebra A we will write $A = R^{[n]}$ to mean that A is R-isomorphic to the polynomial R-algebra in n variables.

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Given $\mathfrak{p} \in \operatorname{Spec} R$, we let $K(\mathfrak{p})$ be the residue field of $\mathfrak{p}$. An R-algebra A is said to be an $\mathbb{A}^n$-fibration if it is finitely generated and flat and for every $\mathfrak{p} \in \operatorname{Spec} R$ we have $K(\mathfrak{p}) \otimes_R A = K(\mathfrak{p})^{[n]}$.

The structure of affine fibrations was studied by Asanuma in [1]. One of the main results is that an $\mathbb{A}^n$-fibration A over a Noetherian ring R is stably trivial, i.e., $A^{[m]} = R^{[n+m]}$ for some $m \geq 1$, if and only if the A-module $\Omega_{A/R}$ of differentials of A over R is stably free. Moreover, in the case $n = 1$ and $\mathbb{Q} \subseteq R$, it follows from the results of Hamann [15] that $A = R^{[1]}$ if and only if $\Omega_{A/R}$ is free, see [5]. In the case $n = 2$, motivated by a question of Freudenburg [14], we studied in [10] fixed point free locally nilpotent derivations of $\mathbb{A}^2$-fibrations over a Noetherian domain containing $\mathbb{Q}$. In particular, a stably trivial $\mathbb{A}^2$-fibration over a factorial ring R containing $\mathbb{Q}$ is trivial if and only if it has a fixed point free locally nilpotent R-derivation. In this paper we generalize this result to an arbitrary ring R containing $\mathbb{Q}$.

2. Notation and preliminaries

In this section we recall the concepts and results to be used in this paper.

2.1. Affine fibrations and residual coordinates

Given a ring R and $\mathfrak{p} \in \operatorname{Spec} R$, the residue field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is denoted by $K(\mathfrak{p})$. Given an R-module M we let $\operatorname{Sym}_R(M)$ be the symmetric algebra of M. For an R-algebra A we let $\Omega_{A/R}$ be the A-module of Kähler differentials of A over R.

Definition 2.1. An R-algebra A is said to be an $\mathbb{A}^n$-fibration if it satisfies the following properties.

1. A is finitely generated as an R-algebra.
2. A is flat as an R-module.
3. For every $\mathfrak{p} \in \operatorname{Spec} R$ we have $K(\mathfrak{p}) \otimes_R A = K(\mathfrak{p})^{[n]}$.

From the third property one easily deduces that the morphism $\operatorname{Spec} A \longrightarrow \operatorname{Spec} R$, induced by the homomorphism $R \longrightarrow A$, is surjective. This property together with the flatness assumption imply that A is faithfully flat over R. In particular, the homomorphism $R \longrightarrow A$ is injective and hence we can view R as a subring of A.

An $\mathbb{A}^n$-fibration A over R is said to be trivial if $A = R^{[n]}$. The fibration is said to be stably trivial if $A^{[m]} = R^{[m+n]}$ for some $m \geq 0$.

Let us now recall the concept of residual coordinate. Usually, it is defined in polynomial rings [4]. But in fact, as noticed in [9], this concept still makes sens if one considers affine fibrations instead of polynomial rings.

Let R be a ring and let A be an $\mathbb{A}^n$-fibration over R. Given $\mathfrak{p} \in \operatorname{Spec} R$, we let $\pi_\mathfrak{p} : A \longrightarrow K(\mathfrak{p}) \otimes_R A$ be the canonical homomorphism. An element $u \in A$ is said