Journal of Algebra 459 (2016) 376–388

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The homogeneous spectrum of Milnor–Witt K-theory

Riley Thornton

Reed College, United States

ARTICLE INFO

Article history:
Received 15 December 2015
Available online 4 May 2016
Communicated by Paul Balmer

MSC:
14F42
19G12
18E30

Keywords:
Milnor–Witt K-theory
Tensor triangular geometry
Stable motivic homotopy theory

ABSTRACT

For any field F (of characteristic not equal to 2), we determine the Zariski spectrum of homogeneous prime ideals in $K_*^{MW}(F)$, the Milnor–Witt K-theory ring of F. As a corollary, we recover Lorenz and Leicht's classical result on prime ideals in the Witt ring of F. Our computation can be seen as a first step in Balmer's program for studying the tensor triangular geometry of the stable motivic homotopy category.

1. Introduction

In this note we completely determine the Zariski spectrum of homogeneous prime ideals in $K_*^{MW}(F)$, the Milnor–Witt K-theory of a field F. This graded ring contains information related to quadratic forms over F — in fact, $K_0^{MW}(F) \cong GW(F)$, the Grothendieck–Witt ring of F — and the Milnor K-theory of F, which appears as a natural quotient of $K_*^{MW}(F)$. While the prime ideals in $GW(F)$ are known classically via a theorem of Lorenz and Leicht [4] (see also [1, Remark 10.2]), we discover a more

E-mail address: rithorn@reed.edu.

http://dx.doi.org/10.1016/j.jalgebra.2016.04.012

refined structure in $\mathrm{Spec}^h(K_*^{MW}(F))$, including a novel class of characteristic 2 primes indexed by the orderings on F which all collapse to the fundamental ideal $I \subseteq GW(F)$ in degree 0.

Much of the interest in $K_*^{MW}(F)$ stems from the distinguished role it plays in Voevodsky's stable motivic homotopy category, $\mathrm{SH}^{\mathbb{A}^1}(F)$. Indeed, a theorem of Morel [7, §6, p. 251] identifies $K_*^{MW}(F)$ with a graded ring of endomorphisms of the unit object in $\mathrm{SH}^{\mathbb{A}^1}(F)$. As $\mathrm{SH}^{\mathbb{A}^1}(F)$ is a tensor triangulated category (with tensor given by smash product, $\wedge$), it may be studied via Balmer's methods of tensor triangular geometry [1]. More specifically, we can look at the full subcategory of compact objects, $\mathrm{SH}^{\mathbb{A}^1}(F)^c$. In this context, the goal is to determine the structure of the triangular spectrum $\mathrm{Spc}(\mathrm{SH}^{\mathbb{A}^1}(F)^c)$ of thick subcategories of $\mathrm{SH}^{\mathbb{A}^1}(F)^c$ which satisfy a "prime ideal" condition with respect to $\wedge$. Balmer's primary tool in the study of triangular spectra is a naturally defined continuous map

$$\rho^\bullet : \mathrm{Spc}(\mathrm{SH}^{\mathbb{A}^1}(F)) \to \mathrm{Spec}^h(K_*^{MW}(F))$$

with codomain the Zariski spectrum of homogeneous prime ideals in $K_*^{MW}(F)$.

By identifying $\mathrm{Spec}^h(K_*^{MW}(F))$, we undertake a first step in Balmer's program for studying the tensor triangular geometry of $\mathrm{SH}^{\mathbb{A}^1}(F)$. In particular, this raises the possibility of studying surjectivity properties of $\rho^\bullet$ (which, in general, are unknown — see [1, Remark 10.5]) by explicitly constructing triangular primes lying over points in $\mathrm{Spec}^h(K_*^{MW}(F))$.

Outline of the paper. The determination of $\mathrm{Spec}^h(K_*^{MW}(F))$ proceeds as follows. Section 2 gives general background on Milnor–Witt K-theory and states our main result. In subsections 3.1 and 3.2, the homogeneous spectra of two quotients of $K_*^{MW}(F)$ are determined, and in subsection 3.3 the two quotient spectra are stitched together to get the full spectrum.

2. Milnor–Witt K-theory

The Milnor–Witt K-theory of a field, $K_*^{MW}(F)$, is a graded ring associated to a field by taking a certain quotient of the free algebra on a symbol η and the set of formally bracketed units in the field as follows:

Definition 2.1. For a set S, $[S] = \{[s] : s \in S\}$ is the set of (purely formal) symbols in S.

The free associative algebra on $[F^\times] \cup \{\eta\}$ is

$$FrAl([F^\times] \cup \{\eta\}) = \left\{ \sum_{1 \le i \le n} a_i \sigma_{i1}, ..., \sigma_{ij_i} \,:\, a_i \in \mathbb{Z},\ \sigma_{ij} \in [F^\times] \cup \{\eta\},\ n \in \mathbb{N} \right\}$$

with multiplication and addition completely determined by the ring axioms.