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# Minimal model of Ginzburg algebras

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ABSTRACT

We compute the minimal model for Ginzburg algebras associated to acyclic quivers $Q$. In particular, we prove that there is a natural grading on the Ginzburg algebra making it formal and quasi-isomorphic to the preprojective algebra in non-Dynkin type, and in Dynkin type is quasi-isomorphic to a twisted polynomial algebra over the preprojective with a unique higher $A_\infty$-composition. To prove these results, we construct and study the minimal model of an $A_\infty$-envelope of the derived category $\mathcal{D}^b(Q)$ whose higher compositions encode the triangulated structure of $\mathcal{D}^b(Q)$.



## Introduction

Calabi–Yau categories have been enjoying a growing interest in geometry, representation theory, and mathematical physics. These are triangulated $k$-categories (where $k$ is an algebraically closed ground field) equipped with bifunctorial isomorphisms

$$\operatorname{Hom}(X, Y) \cong D \operatorname{Hom}(Y, S^n X)$$

where $D$ denotes $k$-linear dual and $S$ is the suspension functor of the triangulated category. Here, $n$ is a fixed integer called the *Calabi–Yau dimension*. Such categories have

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appeared in diverse areas of mathematics in many different guises. They have appeared as everything from derived categories of sheaves on noncommutative spaces, to categories of $D$-branes in string theory, and to cluster mutation in the representation theory of finite dimensional algebras.

The eponymous instance of such categories comes from algebraic geometry: Serre duality implies that the derived category of coherent sheaves for an $n$-dimensional Calabi–Yau variety $X$ is an $n$-Calabi–Yau category. In some sense this is the general situation. By adopting the categorical approach to noncommutative geometry in the sense of Bondal [3], one should view an arbitrary Calabi–Yau category as the derived category of coherent sheaves of some noncommutative Calabi–Yau space.

Dimension three is of particular interest. String theory requires an extra six spatial dimensions taking the form of a complex Calabi–Yau 3-fold. Three-Calabi–Yau categories are an essential ingredient in Kontsevich's homological picture of mirror symmetry [17]. The motivic DT-invariants of Kontsevich and Soibelman are framed as invariants of 3-Calabi–Yau categories [18]. In the direction of representation theory, 3-Calabi–Yau categories appear in Amiot's construction of cluster categories [1], extending the construction of Buan, Marsh, Reiten, Reineke, and Todorov, which themselves are 2-Calabi–Yau [5].

In his seminal work, [8] Ginzburg introduced a class of differential graded algebras called Calabi–Yau algebras which have the remarkable property that their derived categories are Calabi–Yau triangulated categories. For dimension three, one can associate a Ginzburg algebra to an arbitrary quiver with potential in the sense of Derksen, Weyman, and Zelevinsky [6]. Conversely, Keller (and Van den Bergh in the appendix) proved in [16] that the Ginzburg algebra is 3-Calabi–Yau while Van den Bergh proved a converse result in [22].

Despite their growing mathematical importance, surprisingly little has been done in the study of Ginzburg algebras from the point of view of quiver representations. In this paper we study the $A_\infty$-structure of the Ginzburg algebras associated to acyclic quivers and compute their minimal models. In particular, we give a complete description of the $A_\infty$-structure of the minimal models for all acyclic quivers.

### *0.1. Results*

Fix a ground field $k$, which we will assume to be algebraically closed and characteristic 0. Throughout, all categories are assumed to be $k$-categories, all functors are $k$-linear, and all unadorned tensor products are over $k$. For a category $\mathcal{C}$ denote by $\mathcal{C}(X,Y)$ the space of morphisms from $X$ to $Y$ in $\mathcal{C}$.

Recall a *quiver* is a tuple $Q = (Q_0, Q_1, s, t)$ where $Q_0$ is the *vertex set*, $Q_1$ is the *arrow* set, and $s, t : Q_1 \to Q_0$ are respectively the *source* and *target* functions. A *path* in $Q$ is a sequence $\alpha_1\alpha_2\cdots\alpha_n$ of arrows with $t(\alpha_i) = s(\alpha_{i+1})$ for each $1 \leq i < n$.