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Hilbert series of modules over positively graded polynomial rings

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In this note, we give examples of formal power series satisfying certain conditions that cannot be realized as Hilbert series of finitely generated modules. This answers to the negative a question raised in a recent article by the second and the third author. On the other hand, we show that the answer is positive after multiplication with a scalar.

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1. Introduction

Let $\mathbb{K}$ be a field, and let $R = \mathbb{K}[X_1, \ldots, X_n]$ be the positively $\mathbb{Z}$-graded polynomial ring with $\deg X_i = d_i \geq 1$ for every $i = 1, \ldots, n$. Consider a finitely generated graded R-module $M = \bigoplus_k M_k$ over R. The graded components M_k of M are finitely dimensional $\mathbb{K}$-vector spaces, and, since R is positively graded, $M_k = 0$ for $k \ll 0$. The formal Laurent series

$$H_M(t) := \sum_{k \in \mathbb{Z}} (\dim_K M_k) t^k \in \mathbb{Z}[\![t]\!][t^{-1}]$$

is called the Hilbert series of M. Obviously every coefficient of this series is nonnegative. Moreover, it is well-known that $H_M(t)$ can be written as a rational function with denominator $(1-t^{d_1}) \cdots (1-t^{d_n})$. In fact, in the standard graded case (i.e. $d_1 = \cdots = d_n = 1$) these two properties characterize the Hilbert series of finitely generated R-modules among the formal Laurent series $\mathbb{Z}[\![t]\!][t^{-1}]$, cf. Uliczka [4, Cor. 2.3].

In the non-standard graded case, the situation is more involved. A characterization of Hilbert series was obtained by the second and third author in [2]:

Theorem 1.1 *(Moyano–Uliczka). Let $P(t) \in \mathbb{Z}[\![t]\!][t^{-1}]$ be a formal Laurent series which is rational with denominator $(1-t^{d_1}) \cdots (1-t^{d_n})$. Then P can be realized as Hilbert series of some finitely generated R-module if and only if it can be written in the form*

$$P(t) = \sum_{I \subseteq \{1,\ldots,n\}} \frac{Q_I(t)}{\prod_{i \in I} (1-t^{d_i})} \tag{1.1}$$

with nonnegative $Q_I \in \mathbb{Z}[t, t^{-1}]$.

However, it remained an open question in [2, Remark 2.3] if the condition of the Theorem is satisfied by *every* rational function with the given denominator and nonnegative coefficients. In this paper we answer this question to the negative. In Section 3 we provide examples of rational functions that do not admit a decomposition (1.1) and are thus not realizable as Hilbert series. On the other hand, we show the following in Corollary 2.5 and Theorem 2.6:

Theorem 1.2. *Assume that the degrees $d_1, \ldots, d_n$ are pairwise either coprime or equal. Then the following holds:*

1. *If $n = 2$, then every rational function $P(t) \in \mathbb{Z}[\![t]\!][t^{-1}]$ with the given denominator and nonnegative coefficients admits a decomposition as in (1.1).*
2. *In general, the same still holds* up to multiplication with a scalar.

In particular, there is a formal Laurent series $P(t)$ with integral coefficients such that $2P(t)$, but not $P(t)$, is the Hilbert series of a finitely generated graded R-module,