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Quantum automorphism group of the lexicographic product of finite regular graphs

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ABSTRACT

We study the quantum automorphism group of the lexicographic product of two finite regular graphs, providing a quantum generalization of Sabidussi's structure theorem on the automorphism group of such a graph.

1. Introduction

A quantum permutation group on n points is a compact quantum group acting faithfully on the classical space consisting of n points. The following facts were discovered by Wang [16].

(1) There exists a largest quantum permutation group on n points, now denoted S_n^+, and called *the* quantum permutation groups on n points.
(2) The quantum group S_n^+ is infinite-dimensional if $n \geq 4$, and hence in particular an infinite compact quantum group can act faithfully on a finite classical space.

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Very soon after Wang's paper [16], the representation theory of S_n^+ was described by Banica [1]: it is similar to the one of $SO(3)$ and can be described using tensor categories of non-crossing partitions. This description, further axiomatized and generalized by Banica–Speicher [5], led later to spectacular connections with free probability theory, see e.g. [9].

The next natural question was the following one: does S_n^+ have many non-classical quantum subgroups, or is it isolated as an infinite quantum group acting faithfully on a finite classical space?

In order to find quantum subgroups of S_n^+, the quantum automorphism group of a finite graph was defined in [6,2]. This construction indeed produced many examples of non-classical quantum permutation groups, answering positively to the above question. The known results on the computation of quantum symmetry groups of graphs are summarized in [4], where the description of the quantum symmetry group of vertex-transitive graphs of small order (up to 11) is given (with an exception for the Petersen graph, whose quantum automorphism group remains mysterious).

The present paper is a contribution to the study of quantum automorphism groups of finite graphs: we study the quantum automorphism group of a lexicographic product of finite regular graphs, for which we generalize the results from [4]. The description of the quantum automorphism group of some lexicographic product of finite graphs was, amongst other ingredients, a key step in [4] in the description of the quantum automorphism group of small graphs. Recall that if X, Y are finite graphs, their lexicographic product is, roughly speaking, obtained by putting a copy of X at each vertex of Y (see Section 3 for details). There is, in general, a group embedding

$$\mathrm{Aut}(X) \wr \mathrm{Aut}(Y) \subset \mathrm{Aut}(X \circ Y) \quad (*)$$

where the group on the left is the wreath product of $\mathrm{Aut}(X)$ by $\mathrm{Aut}(Y)$. A quantum analogue of the above embedding is given in [4], using the free wreath product from [7] and a sufficient spectral condition was given to ensure that the quantum analogue of the embedding is an isomorphism. However, there exist (vertex-transitive) graphs of order ≥ 12 that do not satisfy the spectral assumption, and for which the embedding $(*)$ is an isomorphism (see Example 4.14), hence the results in [4] are not sufficient to fully understand quantum symmetry groups of lexicographic products.

A necessary and sufficient condition on the graphs X, Y in order that the embedding $(*)$ be an isomorphism was given by Sabidussi in [14] (see Section 4). The conditions look slightly technical at first sight, but are very easy to check in practice. In this paper we provide a quantum generalization of Sabidussi's result: we show that for a pair of regular graphs X, Y, the quantum analogue of the embedding $(*)$ is an isomorphism if and only if the graphs satisfy Sabidussi's conditions: see Theorem 4.5. Our result covers many graphs that do not satisfy the spectral conditions from [4].

As a final comment, we wish to point out that our result, which expresses certain quantum automorphism groups of finite graphs as free wreath products, will be useful