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One-dimensional stable rings

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ABSTRACT

A commutative ring R is stable provided every ideal of R containing a nonzerodivisor is projective as a module over its ring of endomorphisms. The class of stable rings includes the one-dimensional local Cohen–Macaulay rings of multiplicity at most 2, as well as certain rings of higher multiplicity, necessarily analytically ramified. The former are important in the study of modules over Gorenstein rings, while the latter arise in a natural way from generic formal fibers and derivations.

We characterize one-dimensional stable local rings in several ways. The characterizations involve the integral closure $\overline{R}$ of R and the completion of R in a relevant ideal-adic topology. For example, we show: If R is a reduced stable ring, then there are exactly two possibilities for R: (1) R is a *Bass ring*, that is, R is a reduced Noetherian local ring such that $\overline{R}$ is finitely generated over R and every ideal of R is generated by two elements; or (2) R is a *bad stable domain*, that is, R is a one-dimensional stable local domain such that $\overline{R}$ is not a finitely generated R-module.

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1. Introduction

The class of stable rings has a long history dating back at least to the "Ubiquity" paper of Bass, where he showed that rings for which every ideal can be generated by two elements are stable [5, Corollary 7.3]. Following Lipman [19] and Sally and Vasconcelos [32,33], we define an ideal I of a commutative ring R to be *stable* if I is projective over its ring of endomorphisms. A ring R is *stable* if every regular ideal of R (that is, every ideal containing a nonzerodivisor) is stable.[1] Independently of Bass, Lipman [19] studied stable ideals in one-dimensional semilocal Cohen–Macaulay rings and showed how stability was reflected in terms of invariants of the ring such as multiplicity, embedding dimension and the Hilbert function. (We recall the definitions of these terms in Section 2.1.) The terminology of "stable" ideal originates with Lipman; it reflects the stabilization of a certain chain of infinitely near local rings. Lipman in turn was motivated to introduce these ideals as a way to unify ideas from Arf and Zariski involving singularities of plane curves, and to produce the largest ring between a one-dimensional local Cohen–Macaulay ring R and its integral closure having the same multiplicity sequence as R [19, Corollary 3.10].

Sally and Vasconcelos, motivated by the work of Bass and Lipman, studied stable Noetherian local rings in detail in [32,33], and showed that a reduced local Cohen–Macaulay ring with finite normalization is stable if and only if every ideal of R can be generated by two elements, thus substantiating a conjecture of Bass and proving a partial converse to his theorem mentioned above. Reduced Noetherian local rings having finite normalization and every ideal generated by two elements are known in the literature as *Bass rings* because of their importance in Bass's article [5]. Sally and Vasconcelos [33, Example 5.4] gave the first example of a Noetherian stable domain R of multiplicity > 2 (and hence without finite normalization). They used a construction of Ferrand and Raynaud that exhibits R as the preimage of a derivation over a specific field of characteristic 2. Heinzer, Lantz and Shah [14, (3.12)] showed that this technique can be modified to produce over this same field of characteristic 2 for each $e > 2$ an analytically ramified Noetherian local stable domain of multiplicity e. More recently, analytically ramified Noetherian stable local rings have proved to be useful theoretical tools for classifying the rank one discrete valuation rings (DVRs) that arise as the normalization of an analytically ramified one-dimensional Noetherian local domain, as well as for describing properties of the generic formal fiber of a Noetherian local domain [24,25]; see Remark 2.15 and Theorem 2.16.[2]

The main goal of this article is to tie all these results together. The Noetherian assumption is not essential to most of our arguments, and thus we use general ideal-theoretic methods to work in a setting in which R is a one-dimensional local ring with regular

[1] Our definition of a stable ring differs slightly from Sally and Vasconcelos [32,33] in that we require only that every regular ideal is stable, not that every ideal is stable. However, for a reduced one-dimensional Cohen–Macaulay ring, the two definitions agree [28, p. 260].

[2] The definitions of normalization, finite normalization and analytically ramified rings appear at the end of this section and in Section 2.1.