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Torsion and tensor products over domains and specializations to semigroup rings ☆

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ABSTRACT

Let R be a commutative Noetherian domain, and let M and N be finitely generated R-modules. We give new criteria for determining when the tensor product of two modules has torsion. We also give constructive formulas for the torsion submodule of $M \otimes_R N$. In certain cases we determine bounds on the length and minimal number of generators of the torsion submodule. Lastly, we focus on the case where R is a numerical semigroup ring with the goal of making progress on the Huneke–Wiegand Conjecture.

Throughout this paper R will denote a commutative domain. Given an R-module M, the set $\mathrm{T}(M) := \{x \in M \mid rx = 0 \text{ for some } r \neq 0 \text{ in } R\}$ is called the torsion submodule of M. A module M is said to be torsion-free provided that $\mathrm{T}(M) = 0$. Otherwise we say that M has torsion.

Given a ring R and non-projective R-modules M and N, it is common to find that the tensor product, $M \otimes_R N$, has torsion. For instance C. Huneke and R. Wiegand show that when R is a one-dimensional local hypersurface domain, then $M \otimes_R N$ always has torsion, except in the trivial case where either M or N is free and the other is

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torsion-free [6]. However, elementary examples show that their result cannot be extended to complete intersection domains of higher codimension. For instance given a field k and an indeterminant t, the ring $R = k[t^4, t^5, t^6]$ is complete intersection of codimension two, and the module $(t^4R + t^5R) \otimes_R (t^4R + t^6R)$ is torsion-free. A natural question that arises is, for which modules and which classes of rings can one use the assumption that $M \otimes_R N$ is torsion-free as a criterion for projectivity in either M or N. With this in mind we consider the following conjecture:

The Huneke–Wiegand Conjecture. (See [6, 473–474].) If R is a one dimensional Gorenstein local domain such that M and $M \otimes_R \operatorname{Hom}_R(M, R)$ are torsion-free, then M is free.

Additional motivation for focusing on the above conjecture is that a positive answer would also imply the Auslander–Reiten conjecture in the case of Gorenstein domains of arbitrary dimension; see [3, Proposition 5.6] for details.

Although past research on torsion and tensor products has primarily focused on determining when $\mathrm{T}(M \otimes_R N) = 0$, in this paper we take a broader approach to the subject. In some cases we are able to offer a formula for $\mathrm{T}(M \otimes_R N)$ in terms of N and the generators of M using only sums, products, intersections and a single quotient; see for instance Lemma 1.2 and Theorem 1.4. In such instances our constructions naturally lead to new equivalences for when $\mathrm{T}(M \otimes_R N) = 0$. One of these equivalences, Proposition 4.3, has already been cited in order to solve a new case of the Huneke–Wiegand Conjecture and has spurred research on the factorization properties of a previously unstudied class of monoids; see [4] and [5] for details.

In the case where the integral closure of R is a discrete valuation ring, we offer bounds on the length and the minimal number of generators of $\mathrm{T}(M \otimes_R N)$. These bounds are related to the formulas we get in the ideal setting where R is a numerical semigroup ring and M and N are monomial ideals. In the case of monomial ideals over numerical semigroup rings, the torsion submodule is always generated by differences of simple tensors. This construction allows us to reduce the torsion-free property to a more elementary statement about the distributive property with regard to sums and intersections of integral sets; see for instance Remark 4.10.

We begin by focusing on the general case where R is an arbitrary commutative domain, and steadily over the course of the paper the conditions on R become more and more refined. In section one we produce exact constructions for $\mathrm{T}(M \otimes_R N)$, both for the case where M is a two generated ideal and also when M and N are monomial ideals over $\mathbb{Z}^n$ standard graded algebras. We then use these constructions to develop equivalences for when $\mathrm{T}(M \otimes_R N) = 0$. In the second section we focus on the case where the integral closure of R is a discrete valuation ring with valuation v, and the modules M and N are ideals of R. In this case $v(R)$ is a numerical semigroup with relative ideals $v(M)$ and $v(N)$. In this section we produce bounds on the length of $\mathrm{T}(M \otimes_R N)$ with relation to the relative ideals $v(M)$ and $v(N)$. In the process we define tensors of semigroup ideals