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On the type of an almost Gorenstein monomial curve

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ABSTRACT

We prove that the Cohen–Macaulay type of an almost Gorenstein monomial curve $\mathcal{C} \subseteq \mathbb{A}^4$ is at most 3, and make some considerations on the general case.

Introduction

Almost Gorenstein rings have been introduced by Barucci and Fröberg (cf. [3]) as a larger class of Cohen–Macaulay rings that are next to Gorenstein. In the same work, the authors proved some results for this class of rings, that found applications in [7].

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The original definition was given for one-dimensional analytically unramified local rings; however recently Goto et al. (cf. [6]) adapted this definition in order to deal with local Cohen–Macaulay rings of arbitrary dimension.

This work is focused on investigating possible bounds for the Cohen Macaulay type of local rings associated to almost Gorenstein monomial curves, in function of the embedding dimension. It is well-known that for one-dimensional analytically unramified local rings with embedding dimension 3, not necessarily almost Gorenstein, the Cohen Macaulay type does not exceed 2 (cf. [5], Theorem 11). However, in the same paper it has been showed that, if the embedding dimension is greater than 3, there is no upper bound for the type. Thus the smallest interesting case is that of the coordinate ring of an almost Gorenstein monomial curve in $\mathbb{A}^4$. In this setting, further motivation for this work arises from a question by Numata (cf. [9]), which we prove with the following:

Theorem 1. *The Cohen–Macaulay type of an almost Gorenstein monomial curve $\mathcal{C} \subseteq \mathbb{A}^4$ is at most 3.*

Many examples are present in literature (cf. [9]) of almost Gorenstein monomial curves $\mathcal{C} \subseteq \mathbb{A}^4$ with type 3; therefore, this bound is sharp.

The first section of this paper is devoted to proving Theorem 1, while in Section 2 we provide computational evidence and theoretical considerations for higher embedding dimensions. To simplify the exposition we will use the language of numerical semigroups (cf. [10]). Given the correspondence between numerical semigroups and monomial curves (cf. [2]), in order to prove Theorem 1 it suffices to prove that the type of a 4-generated almost symmetric numerical semigroup is at most 3.

1. Main result

Denote by $\mathbb{Z}$ and $\mathbb{N}$ the set of integers and nonnegative integers respectively. Given $e \geq 2$ and $n_1, n_2, \ldots, n_e \in \mathbb{N}$ such that $\gcd(n_1, n_2, \ldots, n_e) = 1$, the *numerical semigroup* generated by $\{n_1, n_2, \ldots, n_e\}$ is the set

$$S = \langle n_1, n_2, \ldots, n_e \rangle = \{a_1 n_1 + a_2 n_2 + \ldots + a_e n_e \mid a_i \in \mathbb{N}\},$$

which is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N} \setminus S$ is finite. With the notation $S = \langle n_1, n_2, \ldots, n_e \rangle$ we will assume that $\{n_1, n_2, \ldots, n_e\}$ is a minimal generating system for S; we will say that e is the *embedding dimension* of S, denoted by $e(S)$. We also denote by $F(S)$ the *Frobenius number* of S, that is, $F(S) = \max \mathbb{Z} \setminus S$, and by $PF(S)$ the set of *pseudo-Frobenius* numbers of S,

$$\begin{aligned} PF(S) &= \{x \notin S \mid x + s \in S \text{ for every } s \in S \setminus \{0\}\} \\ &= \{x \notin S \mid x + n_i \in S \text{ for every } i = 1, \ldots, e\}, \end{aligned}$$

whose cardinality is called the *type* of S, denoted by $t(S)$.