Journal of Algebra 456 (2016) 278–293

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Character varieties of free groups are Gorenstein but not always factorial

Sean Lawton *, Christopher Manon

Department of Mathematical Sciences, George Mason University, 4400 University Drive, Fairfax, VA 22030, USA

ARTICLE INFO

Article history:
Received 13 April 2015
Communicated by Michel Van den Bergh

MSC:
14D20
14L30
16U30
13H10
16E65

Keywords:
Character variety
Moduli space
Free group
Reductive
Gorenstein
UFD

ABSTRACT

Fix a rank g free group F_g and a connected reductive complex algebraic group G. Let $\mathcal{X}(F_g, G)$ be the G-character variety of F_g. When the derived subgroup $DG < G$ is simply connected we show that $\mathcal{X}(F_g, G)$ is factorial (which implies it is Gorenstein), and provide examples to show that when DG is not simply connected $\mathcal{X}(F_g, G)$ need not even be locally factorial. Despite the general failure of factoriality of these moduli spaces, using different methods, we show that $\mathcal{X}(F_g, G)$ is always Gorenstein.

* Corresponding author.
E-mail addresses: slawton3@gmu.edu (S. Lawton), cmanon@gmu.edu (C. Manon).

http://dx.doi.org/10.1016/j.jalgebra.2016.01.044
0021-8693/

1. Introduction

Let Γ be a finitely generated group (generated by g elements), perhaps the fundamental group of a manifold or orbifold M, and let G be a reductive affine algebraic group over an algebraically closed field $\Bbbk$. Then the collection of group homomorphisms $\mathrm{Hom}(\Gamma, G)$ is naturally a $\Bbbk$-variety cut out of the product variety G^g via the relations defining Γ. Since G admits a faithful morphism into $\mathrm{GL}_n(\Bbbk)$, we call $\mathrm{Hom}(\Gamma, G)$ the G-representation variety of Γ. Standard in representation theory is that two representations are equivalent if they are conjugate. As we are interested in G-valued representations, we consider the conjugation action of G on the representation variety. The orbit space $\mathrm{Hom}(\Gamma, G)/G$ is generally not Hausdorff, but it is homotopic (see [6, Proposition 3.4]) to the geometric points of the Geometric Invariant Theory (GIT) quotient $\mathcal{X}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)/\!\!/G$; which as usual, is the spectrum of the ring of G-invariant elements in the coordinate ring $\Bbbk[\mathrm{Hom}(\Gamma, G)]$.

The spaces $\mathcal{X}(\Gamma, G)$, the *G-character varieties of* Γ, constitute a large class of affine varieties (see [29]). More importantly, they are of central interest in the study of moduli spaces (see [1,10,23,32,33]); finding applications in mathematical physics (see [17,37,12,18]), the study of geometric manifolds (see [7]), and knot theory (see [3]).

The purpose of this note is to prove that the singularities of a character variety $\mathcal{X}(F_g, G)$ are always well behaved, when G is reductive and F_g is a rank g free group. This is in stark contrast to the situation when Γ is not free; see [15]. In particular, we characterize when $\mathcal{X}(F_g, G)$ is factorial (in arbitrary characteristic) and show it is always Gorenstein (when $\Bbbk = \mathbb{C}$). This allows us to describe conditions when the Picard group of $\mathcal{X}(F_g, G)$ is trivial.

In general, regular rings (smooth varieties) are local complete-intersection rings (varieties locally cut out by codimension relations) which are Gorenstein. All Gorenstein rings are Cohen–Macaulay, and assuming the singular locus lies in codimension at least 2, this further implies normality (which implies irreducibility). So Gorenstein varieties (coordinate rings are Gorenstein) are situated between complete-intersections and Cohen–Macaulay varieties. Factorial varieties (coordinate rings are unique factorization domains) fall slightly in between. When they are additionally normal, they have the property that all irreducible hypersurfaces (divisors) come from rational functions (principal divisors). A factorial Cohen–Macaulay variety is automatically Gorenstein (but not conversely). For $G = \mathrm{SL}_n(\mathbb{C})$, we describe exactly when these properties hold for $\mathcal{X}(F_g, G)$; see Example 2.7.

Here is the main theorem of the paper:

Theorem 1.1. *For any $g \geqslant 1$ and any connected complex reductive algebraic group G, the space $\mathcal{X}(F_g, G)$ is Gorenstein. If the derived subgroup $DG < G$ is simply connected, then $\mathcal{X}(F_g, G)$ is factorial. If the derived subgroup $DG < G$ is not simply connected, there are examples where $\mathcal{X}(F_g, G)$ is not factorial, nor locally factorial.*