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A characterization of nilpotent nonassociative algebras by invertible Leibniz-derivations ☆

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ABSTRACT

Moens proved that a finite-dimensional Lie algebra over a field of characteristic zero is nilpotent if and only if it has an invertible Leibniz-derivation. In this article we prove the analogous results for finite-dimensional Malcev, Jordan, $(-1,1)$-, right alternative, Zinbiel and Malcev-admissible noncommutative Jordan algebras over a field of characteristic zero. Also, we describe all Leibniz-derivations of semisimple Jordan, right alternative and Malcev algebras.

1. Introduction

The theory of Lie algebras having a nonsingular derivation has a rich history, and is still an active research area. Such Lie algebras appear in many different situations, such as in studies of pro-p groups of finite coclass by Shalev and Zelmanov [39,41] and in the

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problems concerning the existence of left-invariant affine structures on Lie groups (see Burde's survey [6] for details).

In 1955, Jacobson [12] proved that a finite-dimensional Lie algebra over a field of characteristic zero admitting a non-singular (invertible) derivation is nilpotent. The problem of whether the inverse of this statement is correct remained open until work [9], where an example of a nilpotent Lie algebra in which all derivations are nilpotent (and hence, singular) was constructed. For Lie algebras in prime characteristic the situation is more complicated. In that case there exist non-nilpotent Lie algebras, even simple ones, which admit nonsingular derivations [4]. The main examples of nonsingular derivations are periodic derivations. Kostrikin and Kuznetsov [24] noted that a Lie algebra admitting a nondegenerate derivation admits a periodic derivation, that is, a derivation d such that $d^N = id$ for some N, and proved that a Lie algebra admitting a derivation of period N is abelian provided that $N \not\equiv 0 \pmod 6$. Burde and Moens proved that a finite-dimensional complex Lie algebra L admits a periodic derivation if and only if L admits a nonsingular derivation whose inverse is again a derivation if and only if L is hexagonally graded [7]. In the case of positive characteristic p, Shalev proved that if a Lie algebra L admits a nonsingular derivation of order $n = p^s m$, where $(m,p) = 1$ and $m < p^2 - 1$, then L is nilpotent [40]. The study of periodic derivations was continued by Mattarei [28–30].

The study of derivations of Lie algebras led to the appearance of the notion of their natural generalization — a pre-derivation of a Lie algebra, which is a derivation of a Lie triple system induced by that algebra. In [3] it was proved that Jacobson's result is also true in the terms of pre-derivations. Several examples of nilpotent Lie algebras in which pre-derivations are nilpotent were presented in [3,5].

In paper [34] a generalization of derivations and pre-derivations of Lie algebras is defined as a Leibniz-derivation of order k. Moens proved that a finite-dimensional Lie algebra over a field of characteristic zero is nilpotent if and only if it admits an invertible Leibniz-derivation. After that, Fialowski, Khudoyberdiyev and Omirov [13] showed that with the definition of Leibniz-derivations from [34] the similar result for non-Lie Leibniz algebras is not true. Namely, they gave an example of non-nilpotent Leibniz algebra which admits an invertible Leibniz-derivation. In order to extend the results of the paper [34] for Leibniz algebras they introduced the definition of Leibniz-derivation of a Leibniz algebra which agrees with the definition of Leibniz-derivation of a Lie algebra and proved that a finite-dimensional Leibniz algebra is nilpotent if and only if it admits an invertible Leibniz-derivation. It should be noted that there exist non-nilpotent Filippov (n-Lie) algebras with invertible derivations (see [47]). In paper [22] authors showed that the same result holds for alternative algebras (particularly, for associative algebras). Also, in this article an example of nilpotent alternative (non-associative) algebra over a field of positive characteristic possessing only singular derivations was provided.

It is well known that the radicals of finite-dimensional algebras belonging to the classical varieties (such as varieties of Jordan algebras, Lie algebras, alternative algebras,