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On the universal family of Hilbert schemes of points on a surface

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A B S T R A C T

For a smooth quasi-projective surface X and an integer $n \geq 3$, we show that the universal family Z^n over the Hilbert scheme $\mathrm{Hilb}^n(X)$ of n points has non-$\mathbb{Q}$-Gorenstein, rational singularities, and that the Samuel multiplicity μ at a closed point on Z^n can be computed in terms of the dimension of the socle. We also show that $\mu \leq n$.
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1. Introduction

Let X be a smooth quasi-projective surface over an algebraically closed field k of characteristic 0. Let $\mathrm{Hilb}^n(X)$ denote the Hilbert scheme of zero dimensional closed subschemes of X of length n. Fogarty's fundamental result [5] claims that $\mathrm{Hilb}^n(X)$ is a smooth, irreducible variety of dimension $2n$. When X is complete, one may consider $\mathrm{Hilb}^n(X)$ as a natural compactification of the space of n unlabelled distinct points on X. For this reason, $\mathrm{Hilb}^n(X)$ is a quite useful tool for dealing with infinitely near points on X.

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The universal family $Z^n \subset \mathrm{Hilb}^n(X) \times X$, with the induced projection $\pi : Z^n \to \mathrm{Hilb}^n(X)$, is a finite flat cover of degree n over $\mathrm{Hilb}^n(X)$. The importance of Z^n lies in its role in inductive approaches for the study of Hilbert schemes, e.g. in deduction of the Picard scheme of $\mathrm{Hilb}^n(X)$ [6], and in calculation of the Nakajima constants [4]. While Z^2 is simply the blow up of $X \times X$ along the diagonal, Z^n is complicated in general. It is well known that Z^n is irreducible, singular (except $n = 2$), and Cohen–Macaulay. By [6], Z^n is also normal with R_3 condition. More precisely, we prove in this note

Theorem 1.1. *The universal family Z^n is non-$\mathbb{Q}$-Gorenstein, and has rational singularities. For a closed point $\zeta = (\xi, p) \in Z^n$, the Samuel multiplicity $\mu = \binom{b_2+1}{2}$, where $b_2 = b_2(\mathcal{O}_{\xi,p})$ is the dimension of the socle of $\mathcal{O}_{\xi,p}$.*

Recall that given a Noetherian local ring $(A, \mathfrak{m})$, there exists a Hilbert–Samuel polynomial $P_A(t) \in \mathbb{Q}[t]$ such that $P_A(l) =$ the length of $A/\mathfrak{m}^l$ for $l \gg 0$. The *Samuel multiplicity* μ_A is defined to be $(\dim A)!$ times the leading coefficient of $P_A(t)$, which measures the asymptotical growth rate of the length of $A/\mathfrak{m}^l$ as l goes to infinity. We note that $b_2 + 1$ equals the minimal number of generators of the ideal $\mathscr{I}_{\xi,p} \subset \mathcal{O}_{X,p}$; see Lemma 2.1.

For each $i > 0$, let $V^i = \{(\xi, p) \in Z^n \mid b_2(\mathcal{O}_{\xi,p}) = i\}$. The locally closed subsets V^i form a stratification of Z^n. A proposition due to Ellingsrud and Lehn [3] claims that $\mathrm{codim}(V^i, \mathrm{Hilb}^n(X) \times X) \geq 2i$. This kind of codimension estimate is an ingredient in proving the irreducibility of $\mathrm{Hilb}^n(X)$ and more generally, that of certain quot schemes. An immediate consequence of their estimate is that $b_2 \leq n + 1$. In fact, we have

Theorem 1.2.

$$b_2 \leq \left\lfloor \frac{\sqrt{1 + 8n} - 1}{2} \right\rfloor,$$

and the bound is optimal. Consequently, one has $\mu \leq n$.

Haiman [8, Prop. 3.5.3] proves a similar result on multiplicity, namely $\dim_k \mathcal{O}_{\xi,p} \geq \binom{b_2+1}{2}$, which implies that $\mu \leq n$ as well. Here we need to point out that $\mu \neq \dim_k \mathcal{O}_{\xi,p}$ in general.

2. Preliminaries

For a point $p \in X$, $\mathfrak{m}_p$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,p}$ and $k(p)$ denotes the residue field. Let $\xi \subset X$ be a zero dimensional closed subscheme with the defining ideal $\mathscr{I}_\xi$. The socle $\mathrm{Soc}\,(\mathcal{O}_{\xi,p})$ is defined to be $\mathrm{Hom}_{\mathcal{O}_{X,p}}(k(p), \mathcal{O}_{\xi,p})$. We denote the minimal number of generators of $\mathscr{I}_{\xi,p}$ by $e(\mathscr{I}_{\xi,p})$. By Nakayama lemma $e(\mathscr{I}_{\xi,p}) = \dim_{k(p)} \mathscr{I}_{\xi,p} \otimes k(p)$.

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