Journal of Algebra 455 (2016) 235–250

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Free algebras and free groups in Ore extensions and free group algebras in division rings [☆]

Jason P. Bell [a,*], Jairo Z. Gonçalves [b]

[a] Department of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

[b] Department of Mathematics, University of São Paulo, São Paulo, 05508-090, Brazil

ARTICLE INFO

Article history:
Received 9 September 2015
Available online 3 March 2016
Communicated by Louis Rowen

MSC:
16K40
16A06
14J50

Keywords:
Division rings
Free groups
Free algebras
Ore extensions
Automorphisms
Derivations
Solvable groups

ABSTRACT

Let K be a field of characteristic zero, let σ be an automorphism of K and let δ be a σ-derivation of K. We show that the division ring $D = K(x; \sigma, \delta)$ either has the property that every finitely generated subring satisfies a polynomial identity or D contains a free algebra on two generators over its center. In the case when K is finitely generated over a subfield k we then see that for σ a k-algebra automorphism of K and δ a k-linear derivation of K, $K(x; \sigma)$ having a free subalgebra on two generators is equivalent to σ having infinite order, and $K(x; \delta)$ having a free subalgebra is equivalent to δ being nonzero. As an application, we show that if D is a division ring with center k of characteristic zero and D^* contains a solvable subgroup that is not locally abelian-by-finite, then D contains a free k-algebra on two generators. Moreover, if we assume that k is uncountable, without any restrictions on the characteristic of k, then D contains the k-group algebra of the free group of rank two.

☆ The first author was supported by NSERC Grant 326532-2011. The second author was supported by Grant CNPq 301.320/2011-0, and FAPESP-Brazil, Proj. Tematico 2009/52665-0.

* Corresponding author.
E-mail addresses: jpbell@uwaterloo.ca (J.P. Bell), jz.goncalves@usp.br (J.Z. Gonçalves).

http://dx.doi.org/10.1016/j.jalgebra.2016.02.011

1. Introduction

There has been much study on free subobjects in division rings, both in the case of free subalgebras [3,4,7,9,19,21,22,24,25,28] and in the case of free subgroups and free subsemigroups of their multiplicative groups [5,11,13,18,23,27]. (See also the survey [15].) In this paper, we study the question of when division algebras formed from Ore extensions of fields contain free subalgebra in two generators over their centers. We recall that given a ring R with an automorphism σ of R and a σ-derivation, which is a map $\delta : R \to R$ satisfying $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$, we can form an *Ore extension*, $R[x; \sigma, \delta]$, of R, which as a set is just the polynomial ring $R[x]$, but which is endowed with multiplication given by $xr = \sigma(r)x + \delta(r)$ for $r \in R$. We will only be concerned with the case where R is a noetherian domain, usually a field, and in this case $R[x; \sigma, \delta]$ is a noetherian domain as well whose classical division ring of quotients will be denoted $D(x; \sigma, \delta)$, where $D = Q(R)$ is the division ring of fractions of R.

Makar-Limanov gave the first result in this direction by showing that if $A = k\{x, y\}/(xy - yx - 1)$ is the Weyl algebra over a field k of characteristic 0, then its quotient division ring contains a free k-algebra on two generators [22]. Since the first Weyl algebra can be realized as a skew extension $k[x][y; \delta]$ where δ is a derivation of $k[x]$ with $\delta(x) = 1$, we see that this fits into the framework of Ore extensions.

The belief is that this phenomenon of division rings containing free subalgebras holds very generally, and should only fail to hold when there is an obvious obstruction, such as being commutative or being algebraic over its center. We call the *free subalgebra conjecture* the statement that a division algebra D must contain a free algebra on two generators over its center unless D is locally PI, that is, all of its affine subalgebras are polynomial identity algebras. This conjecture (in some form) was formulated independently by both Makar-Limanov and Stafford. We refer the reader to [3] and [15] for a more detailed discussion of the conjecture, and past work on the subject. We note that since the time of these papers, there has been additional interesting work of Smith [29,30] on free algebras in certain Ore extensions.

Unfortunately, the free subalgebra conjecture is considered to be a very hard problem. Indeed, it is closely related to the notoriously difficult analogue of the Kurosh problem for division rings, which asks whether division rings that are finitely generated and algebraic over their centers, but that are not finite-dimensional over them, can exist. This Kurosh analogue has remained open for some time, and a proof of the free subalgebra conjecture would give a solution to the Kurosh problem as well. In the case of free subgroups of the multiplicative group of a division ring more is known. In fact, Chiba [5] has shown that if the base field is uncountable then a division ring that is not commutative has a free subgroup inside its multiplicative group. For free algebras, however, the results appear to be harder to obtain.

The first author and Rogalski showed [3] that the free subalgebra conjecture holds for the quotient division rings of iterated Ore extensions of PI rings, if the base field k is uncountable, or if the base field is countable and the associated field automorphism of