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The monomial conjecture and order ideals II

S.P. Dutta

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, United States

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A B S T R A C T

Let I be an ideal of height d in a regular local ring $(R, m, k = R/m)$ of dimension n and let Ω denote the canonical module of R/I. In this paper we first prove the equivalence of the following: the non-vanishing of the edge homomorhpism $\eta_d : \mathrm{Ext}_R^{n-d}(k, \Omega) \to \mathrm{Ext}_R^n(k, R)$, the validity of the order ideal conjecture for regular local rings, and the validity of the monomial conjecture for all local rings. Next we prove several special cases of the order ideal conjecture/monomial conjecture.

1. Introduction

This paper is a sequel to our work in [9]. Let us first state the two conjectures that would constitute the central theme of this article.

Monomial conjecture (M. Hochster.) Let (R, m) be a local ring of dimension n and let $x_1, x_2, \ldots, x_n$ be a system of parameters of R. Then, for every $t > 0$, $(x_1 x_2 \cdots x_n)^t \notin (x_1^{t+1}, \ldots, x_n^{t+1})$.

E-mail address: s-dutta@illinois.edu.

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Order ideal conjecture (Due to Evans and Griffith.) Let (R, m) be a local ring. Let M be a finitely generated module of finite projective dimension over R and let $\mathrm{Syz}^i(M)$ denote the i-th syzygy of M in a minimal free resolution $F_\bullet$ of M over R. If α is a minimal generator of $\mathrm{Syz}^i(M)$, then the ideal $\mathcal{O}_{\mathrm{Syz}^i(M)}(\alpha)$, generated by entries of α in F_{i-1}, has grade greater than or equal to i for every $i > 0$.

We note that the validity of the assertion for $\mathcal{O}_{\mathrm{Syz}^i(M)}(\alpha)$ as defined above is equivalent to the validity of the assertion that the order ideal for $\alpha (= \{f(\alpha) | f \in \mathrm{Hom}_R(\mathrm{Syz}^i(M), R)\})$ has grade greater than or equal to i for every $i > 0$.

Both these conjectures are open in mixed characteristic. A brief account of these and other equivalent conjectures is provided towards the end of this introduction. In this paper we focus our attention on several aspects of the order ideal conjecture over regular local rings and its connection with the monomial conjecture. As a consequence of the order ideal conjecture Bruns and Herzog [2] observed the following:

Let $(R, m, k = R/m)$ be a regular local ring and let I be an ideal of height $d \geq 0$ with a minimal set of generators $\{x_i\}_{1 \leq i \leq t}$. Let $K_\bullet(\underline{x}, R)$ denote the Koszul complex corresponding to $\{x_i\}$. Then $\theta_i : K_i(\underline{x}; R) \otimes k \to \mathrm{Tor}_i^R(R/I, k)$ are 0-maps for $i > d$.

In one of our main results [9, Corollary 1.4] we proved that *if, for any $d \geq 0$ and for any almost complete intersection ideal $I = (x_1, \ldots, x_d, x_{d+1})$ of height d in a regular local ring R, $\theta_{d+1} : K_{d+1}(\underline{x}; R) \otimes k \to \mathrm{Tor}_{d+1}^R(R/I, k)$ is the 0-map, then the monomial conjecture is valid for all local rings.*

Equivalently, we showed that *if the canonical module Ω of R/I is such that $\mathrm{Syz}^d(\Omega)$ has a free summand for $d > 0$, then the monomial conjecture is valid* [9, Section 1.6]. The non-vanishing of θ_{d+1} for almost complete intersection ideals refers to a very special case of the consequence of the order ideal conjecture. In the light of the significance of this special case mentioned above, the following question emerged: How "special" is this special case? We are now able to answer this question in the first theorem of this paper.

Theorem 2.1. *The following statements are equivalent.*

(1) *The order ideal conjecture is valid over any regular local ring R.*
(2) *For every ideal I of R of height $d > 0$, the d-th syzygy in a minimal free resolution of the canonical module of R/I has a free summand.*
(3) *The same statement as in* (2) *for every prime ideal of R.*
(4) *For every almost complete intersection ideal I of height $d > 0$, with $I = (x_1, \ldots, x_{d+1})$, the map $\theta_{d+1} : K_{d+1}(\underline{x}; R) \otimes k \to \mathrm{Tor}_{d+1}^R(R/I, k)$ is the 0-map.*

Thus, the very special case of a consequence of the order ideal conjecture, as mentioned above in (4), implies the order ideal conjecture over regular local rings.

For our next theorem, we need the following set up. Let (R, m, k) be a regular local ring of dimension n and let I be an ideal of height d. From the associativity property of the homomorphism and tensor product, we obtain two spectral sequences with