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Groups having a faithful irreducible representation

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ABSTRACT

We address the problem of finding necessary and sufficient conditions for an arbitrary group, not necessarily finite, to admit a faithful irreducible representation over an arbitrary field.

1. Introduction

We are interested in the problem of finding necessary and sufficient conditions for a group to have a faithful irreducible linear representation. Various criteria have been found when the group in question is finite, as described in §2, so we will concentrate mainly on infinite groups.

Let G be an arbitrary group. Recall that $\mathrm{Soc}(G)$ is the subgroup of G generated by all minimal normal subgroups of G. It is perfectly possible for $\mathrm{Soc}(G)$ to be trivial. We denote by $\mathrm{S}(G)$, $\mathrm{T}(G)$ and $\mathrm{F}(G)$ the subgroups of $\mathrm{Soc}(G)$ generated by all minimal normal subgroups of G that are non-abelian, torsion abelian and torsion-free abelian,

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respectively. For each prime p, let $\mathrm{T}(G)_p$ be the p-part of $\mathrm{T}(G)$. It is a vector space over F_p. Let $\Pi(G)$ be the set of all primes p such that $\mathrm{T}(G)_p$ is non-trivial.

One readily verifies that $\mathrm{T}(G)\,\mathrm{F}(G)$ is abelian and throughout the paper we view $\mathrm{T}(G)\,\mathrm{F}(G)$ as a $\mathbb{Z}G$-module, with G acting by conjugation. Thus, a minimal normal subgroup of G contained in $\mathrm{T}(G)\,\mathrm{F}(G)$ is nothing but an irreducible $\mathbb{Z}G$-submodule of $\mathrm{T}(G)\,\mathrm{F}(G)$.

A normal subgroup N of G is said to be essential if every non-trivial normal subgroup of G intersects N non-trivially.

With this level of generality, it seems unavoidable that the desired necessary and sufficient conditions be stated separately. Our main results are as follows.

Theorem 1.1. *Let G be a group and let K be a field such that:*

(1) $\mathrm{Soc}(G)$ *is essential;*
(2) *If K has prime characteristic p and M is a minimal normal subgroup of G as well as a non-abelian p-group that is not finitely generated, then M admits a non-trivial irreducible representation over K.*
(3) $\mathrm{char}(K) \notin \Pi(G)$*;*
(4) $\mathrm{T}(G)$ *has a subgroup S such that $\mathrm{T}(G)/S$ is locally cyclic and* $\mathrm{core}_G(S) = 1$.

Then G has a faithful irreducible representation over K.

Although condition (1) plays a critical role for us, it is certainly not necessary, as illustrated by the free group $F\{x, y\}$ on 2 generators. It has no minimal normal subgroups and can be faithfully represented, e.g. via $x \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, as in irreducible subgroup of $\mathrm{GL}_2(\mathbb{C})$ (cf. [17, §2.1]).

In connection to condition (2), it is an open question whether a non-abelian simple p-group exists lacking non-trivial irreducible representations in prime characteristic p. The answer is known to be negative under the additional assumption that the group be finitely generated, as shown in [15, Theorem 6.3]. Regardless of the outcome of this question, let $M = M(\mathbb{Q}, \leq, F_p)$ be the McLain group [11], where $\leq$ is the usual order on the rational field $\mathbb{Q}$ and p is a prime. Then M is a minimal normal subgroup of $G = M \rtimes \mathrm{Aut}(M)$ (in fact, $M = \mathrm{Soc}(G)$) as well as a non-abelian p-group that is not finitely generated. Moreover, the only irreducible F_pM-module is the trivial one (since M is a locally finite p-group). Groups like G lie beyond the scope of Theorem 1.1. Perhaps surprisingly, $M \rtimes \mathrm{Aut}(M)$ does have a faithful irreducible representation over F_p (see [20] for details).

In regards to the necessity of conditions (3) and (4), we have the following result.

Theorem 1.2. *Let G be a group such that $[G : C_G(N)] < \infty$ for every minimal normal subgroup N of G contained in $T(G)$. Let K be a field and suppose that G admits a faithful*