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# A Littlewood–Richardson rule for the Macdonald inner product and bimodules over wreath products

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ABSTRACT

We prove a Littlewood–Richardson type formula for $(s_{\lambda/\mu}, s_{\nu/\kappa})_{t^k,t}$, the pairing of two skew Schur functions in the Macdonald inner product at $q = t^k$ for positive integers $k$. This pairing counts graded decomposition numbers in the representation theory of wreath products of the algebra $\mathbb{C}[x]/x^k$ and symmetric groups.

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## 1. Introduction

Let $\Lambda$ denote the algebra of symmetric functions, endowed with the standard bilinear form with respect to which the Schur basis $\{s_\lambda\}$ is orthonormal. The Littlewood–Richardson rule gives an enumerative formula for the inner products

$$c^{\lambda}_{\mu\nu} = (s_\lambda, s_\mu s_\nu) = (s_\nu^\perp s_\lambda, s_\mu),$$

which are known as Littlewood–Richardson coefficients. Here $s_\nu^\perp$ is the linear operator on symmetric functions adjoint to multiplication by the Schur function $s_\nu$. (We refer to [3,5]

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for detailed treatments of the Littlewood–Richardson rule.) The $c^{\lambda}_{\mu\nu}$ are non-negative integers, and they enumerate tableaux satisfying certain conditions. The integrality of the Littlewood–Richardson coefficients is also manifest in their appearance as tensor product multiplicities in the representation theory of $GL(n)$ and as intersection numbers in the Schubert calculus of Grassmannians. A generalization of the Littlewood–Richardson rule adds a fourth partition to the picture: consider the algebra $\mathcal{H}_\Lambda$ of operators on symmetric functions generated by the operators $\{s_\mu, s_\kappa^\perp\}$, which is spanned as a vector space by $\{s_\mu s_\kappa^\perp\}_{\mu,\kappa}$. Define $c^{\kappa\lambda}_{\mu\nu}$ to be the structure constants in the expansion

$$s_\nu^\perp s_\lambda = \sum_{\mu,\kappa} c^{\kappa\lambda}_{\mu\nu} s_\mu s_\kappa^\perp. \tag{1}$$

For $\kappa = \emptyset$ these are the ordinary Littlewood–Richardson coefficients; an enumerative formula for the general case was found by Zelevinsky [7] in the language of *pictures*. The algebra $H_\Lambda$ is a Hopf algebra; in fact it is the Heisenberg double of the Hopf algebra $\Lambda$. Thus the Littlewood–Richardson coefficients may also be thought of as structure constants in the canonical basis of the Hopf algebra $H_\Lambda$.

Let $\Lambda_{q,t}$ denote the algebra of symmetric polynomials over the two-variable coefficient ring $\mathbb{C}(q,t)$, together with the Macdonald inner product $(\,,\,)_{q,t}$, which specializes to the standard inner product at $q = t$. We will be interested in the specialization $\Lambda_{t^k,t}$ for some integer $k \geq 1$. The ring with inner product $\Lambda_{t^k,t}$ appears in several other mathematical contexts, including the representation theory of quantum affine algebras. In particular, several important bases of $\Lambda_{t^k,t}$, such as the Schur basis, should be related to important bases in the representation theory of quantum affine algebras and in the geometry of quiver varieties of affine type. As a result, it is natural to suspect that much of the positive integral structure appearing the ordinary theory of symmetric functions will admit an interesting generalization from $\Lambda$ to $\Lambda_{t^k,t}$. The first goal of the present paper is to extend (1) to the ring $\Lambda_{t^k,t}$, in which the dual is now taken with respect to $(\,,\,)_{t^k,t}$ instead of the standard inner product. As the inner product on $\Lambda_{t^k,t}$ takes values in the ring $\mathbb{C}(t)$, the precise statement involves a $t$-weighted count of tableaux. In Section 2, we define $k$-tableaux, which are fillings of a Young diagram with entries which are monomials of the form $at^m$, $0 \leq m < k$. A $k$-tableau $T$ has an associated statistic $c(T)$, the degree of the product of all its entries. Our main theorem then states:

**Theorem A.** *We have*

$$s_\nu^* s_\lambda = \sum_{\mu,\kappa} c^{\kappa\lambda}_{\mu\nu}(t) s_\mu s_\kappa^*, \quad c^{\kappa\lambda}_{\mu\nu}(t) = \sum_T c(T).$$

Here $c(T)$ is the $t$-degree of the tableau $T$, which ranges over $k$-tableaux, and $f^*$ is the adjoint of the multiplication operator with respect to the Macdonald inner product at $q = t^k$. When $k = 1$, this theorem reduces to the extended Littlewood–Richardson rule of Zelevinsky [7]. If we then set $\kappa$ to the empty partition, the above statement

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