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Nilpotent orbits of certain simple Lie algebras over truncated polynomial rings [☆]

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ABSTRACT

Let $\mathbb{F}$ be an algebraically closed field of positive characteristic $p > 3$, and $\mathfrak{A}$ the divided power algebra in one indeterminate, which, as a vector space, coincides with the truncated polynomial ring of $\mathbb{F}[T]$ by T^{p^n}. Let $\mathfrak{g}$ be the special derivation algebra over $\mathfrak{A}$ which is a simple Lie algebra, and additionally non-restricted as long as $n > 1$. Let $\mathscr{N}$ be the nilpotent cone of $\mathfrak{g}$, and $G = \mathrm{Aut}(\mathfrak{g})$, the automorphism group of $\mathfrak{g}$. In contrast with only finitely many nilpotent orbits in a classical simple Lie algebra, there are infinitely many nilpotent orbits in $\mathfrak{g}$. In this paper, we parameterize all nilpotent orbits, and obtain their dimensions. Furthermore, the nilpotent cone $\mathscr{N}$ is proven to be reducible and not normal. There are two irreducible components in $\mathscr{N}$. The dimension of $\mathscr{N}$ is determined.

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1. Preliminaries

The theory on nilpotent orbits of a reductive Lie algebra $\mathfrak{L} = \mathrm{Lie}(L)$ for a connected reductive group L plays a crucial important role in the structure and representations of $\mathfrak{L}$, especially in the associated geometric aspects. As is well-known, there are finitely many nilpotent orbits in $\mathfrak{L}$. This reduces the study of representations of $\mathfrak{L}$ to a "finite" problem (cf. [4, Theorem 2], [1, Theorems 3.2, 8.5]). The classification theorem for finite-dimensional simple Lie algebras over an algebraically closed field of prime characteristic $p > 5$ implies that each finite-dimensional simple Lie algebra is either of classical type (i.e. one arising from a simple algebraic group) or of Cartan type (cf. [8]). It is a highly-expected task to understand the structure of nilpotent orbits in simple Lie algebras of Cartan type under their adjoint algebraic group action. We initiated the study of this problem in [13] where we precisely gave a structure presentation of nilpotent orbits in the simplest restricted Lie algebra of Cartan type, i.e., the so-called Witt algebra $W(1;1)$. The main results in [13] implies that there are infinitely many nilpotent orbits in $W(1;1)$. A set of representatives of nilpotent orbits in $W(1;1)$, as well as their dimensions, were precisely given. The present work is a sequel to [13]. In this paper, we study the nilpotent orbits in the simplest non-restricted simple Lie algebras, which are special derivation algebras over truncated polynomial rings.

Let us introduce the setup first. We always assume that the ground field $\mathbb{F}$ is algebraically closed, and of prime characteristic $p > 3$, with its prime field denoted by $\mathbb{F}_p$. Let $n \in \mathbb{N}_+$ and $\mathfrak{A} = \mathfrak{A}(1;n) = \mathbb{F}[T]/(T^{p^n})$ be the divided power algebra in one indeterminate, where (T^{p^n}) denotes the ideal of $\mathbb{F}[T]$ generated by T^{p^n}. Set $x^{(i)}$ to be the image of T^i under the quotient map from $\mathbb{F}[T]$ to $\mathfrak{A}$ for $0 \le i \le p^n - 1$. Then there is a canonical basis $\{x^{(0)} = 1, x^{(1)}, \cdots, x^{(p^n-1)}\}$ in $\mathfrak{A}$. In the following, we always write $x^{(1)}$ as x for brevity. The multiplication structure in $\mathfrak{A}$ is defined for $0 \le i, j \le p^n - 1$ via

$$x^{(i)} \cdot x^{(j)} = \begin{cases} \binom{i+j}{i} x^{(i+j)}, & \text{if } i + j < p^n, \\ 0, & \text{if } i + j \ge p^n. \end{cases}$$

It is obvious that $\mathfrak{A}$ is a local algebra with a unique maximal ideal $\mathfrak{m}$ generated by $x^{(p^i)}$ with $0 \le i \le n-1$. It should be noted that the divided power algebra $\mathfrak{A}$ is isomorphic to the truncated polynomial algebra $\mathbb{F}[X]/(X^{p^n})$ as vector spaces. However, they are not isomorphic as algebras in general. More precisely, they are isomorphic as algebras if and only if $n = 1$.

A derivation D over $\mathfrak{A}$ is called special if $D(x^{(i)}) = x^{(i-1)} D(x)$ for $1 \le i \le p^n - 1$. Set $\mathfrak{g}$ to be the space consisting of all special derivations over $\mathfrak{A}$. Then $\mathfrak{g}$ is a simple Lie algebra (cf. [9]), and furthermore $\mathfrak{g} = \mathfrak{A}\partial$ with ∂ being the linear special derivation on $\mathfrak{A}$ subject to the rule $\partial x^{(i)} = x^{(i-1)}$ for $1 \le i \le p^n - 1$, which is generally denoted by $W(1;n)$ and called the Zassenhaus algebras (cf. [9, §4.2]). When $n = 1$, $\mathfrak{g}$ coincides with the aforementioned Witt algebra $W(1;1)$.