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Extension theory and the calculus of butterflies

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ABSTRACT

This paper provides a unified treatment of two distinct viewpoints concerning the classification of group extensions: the first uses weak monoidal functors, the second classifies extensions by means of suitable H^2-actions. We develop our theory formally, by making explicit a connection between (non-abelian) G-torsors and fibrations. Then we apply our general framework to the classification of extensions in a semi-abelian context, by means of butterflies [1] between internal crossed modules. As a main result, we get an internal version of Dedecker's theorem on the classification of extensions of a group by a crossed module. In the semi-abelian context, Bourn's intrinsic Schreier–Mac Lane extension theorem [13] turns out to be an instance of our Theorem 6.3. Actually, even just in the case of groups, our approach reveals a result slightly more general than classical Schreier–Mac Lane theorem.

1. Introduction

Let K and Y be groups. It is well known that the set of (equivalence classes of) split extensions of Y by K is in bijection with the set of Y-actions on K. One way of realizing

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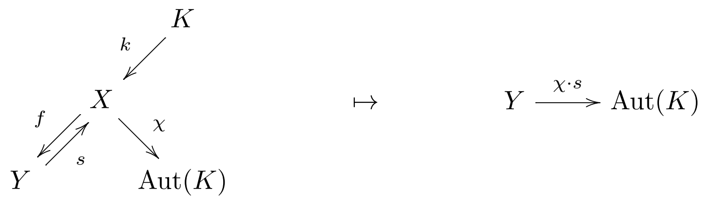
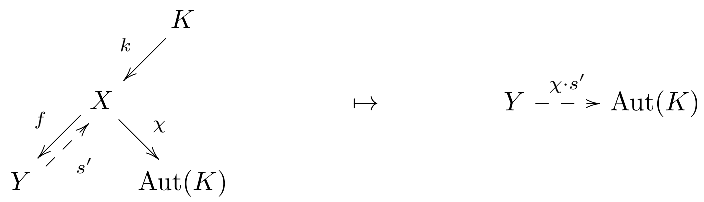
this bijection consists in considering a homomorphic section s of f, and then composing with the canonical conjugation action of X on its normal subgroup K, denoted by χ in the diagram below:



When the extension $K \xrightarrow{k} X \xrightarrow{f} Y$ is no longer split, the homomorphism s fails to exist. Still, since f is surjective, one can find a set-theoretical section s' of f, and consider the composite $\chi \cdot s'$:



However, in this case $\chi \cdot s'$ is no longer an action, in general.

The group $\mathrm{Aut}(K)$ determines the internal groupoid in **Gp**

$$\mathrm{AUT}(K) \qquad = \qquad \begin{array}{c} K \rtimes \mathrm{Aut}(K) \\ d \downarrow \uparrow \downarrow c \\ \mathrm{Aut}(K) \end{array}$$

and the map $\chi \cdot s'$ underlies a (possibly weak) monoidal functor

$$D(Y) \to \mathrm{AUT}(K)\,,$$

where $D(Y)$ is the discrete internal groupoid associated with Y. In other words, $\chi \cdot s'$ is the object map of a functor between the underlying groupoids in **Set**. Notice that different choices of s' give rise to different but isomorphic monoidal functors. This way, we extend the equivalence between split extensions and actions

$$\mathrm{SPLEXT}(Y, K) \simeq \mathbf{Gp}(Y, \mathrm{Aut}(K))$$

to the equivalence

$$\mathrm{EXT}(Y, K) \simeq \mathbf{2Gp}(D(Y), \mathrm{AUT}(K))\,,$$