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Degree three unramified cohomology groups [☆,☆☆]

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A B S T R A C T

Let k be any field, G be a finite group. Let G act on the rational function field $k(x_g : g \in G)$ by k-automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$. Denote by $k(G) = k(x_g : g \in G)^G$, the fixed subfield. Noether's problem asks whether $k(G)$ is rational (= purely transcendental) over k. The unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G))$ and the unramified cohomology $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ are obstructions to the rationality of $\mathbb{C}(G)$ (see [14] and [5]). Peyre proves that, if p is an odd prime number, then there is a group G such that $|G| = p^{12}$, $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) = \{0\}$, but $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq \{0\}$; thus $\mathbb{C}(G)$ is not stably $\mathbb{C}$-rational [12]. Using Peyre's method, we are able to find groups G with $|G| = p^9$ where p is an odd prime number such that $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) = \{0\}$, $H^3_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq \{0\}$.

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1. Introduction

Let k be a field, and L be a finitely generated field extension of k. L is called k-rational (or rational over k) if L is purely transcendental over k, i.e. L is isomorphic to some rational function field over k. L is called stably k-rational if $L(y_1, \ldots, y_m)$ is k-rational for some $y_1, \ldots, y_m$ which are algebraically independent over L. L is called k-unirational if L is k-isomorphic to a subfield of some k-rational field extension of k. It is easy to see that "k-rational" $\Rightarrow$ "stably k-rational" $\Rightarrow$ "k-unirational".

A classical question, the Lüroth problem by some people, asks whether a k-unirational field L is necessarily k-rational. For a survey of the question, see for example [10,6,17].

Noether's problem is a special case of the above Lüroth problem. Let k be a field and G be a finite group. Let G act on the rational function field $k(x_g : g \in G)$ by k-automorphisms defined by $h \cdot x_g = x_{hg}$ for any $g, h \in G$. Denote by $k(G)$ the fixed subfield, i.e. $k(G) = k(x_g : g \in G)^G$. Noether's problem asks, under what situation, the field $k(G)$ is k-rational.

Noether's problem is related to the inverse Galois problem, to the existence of generic G-Galois extensions over k, and to the existence of versal G-torsors over k-rational field extensions [16,13], [7, Section 33.1, page 86].

The first counter-example to Noether's problem was constructed by Swan: $\mathbb{Q}(C_p)$ is not $\mathbb{Q}$-rational if $p = 47$, 113 or 233, etc., where C_p is the cyclic group of order p. Noether's problem for finite abelian groups was studied extensively by Swan, Voskresenskii, Endo and Miyata, Lenstra, etc. For details, see Swan's survey paper [16].

In [14], Saltman defines $\mathrm{Br}_{\mathrm{nr},k}(k(G))$, the unramified Brauer group of $k(G)$ over k. It is known that, if $k(G)$ is stably k-rational, then the natural map $\mathrm{Br}(k) \to \mathrm{Br}_{\mathrm{nr},k}(k(G))$ is an isomorphism; in particular, if k is algebraically closed, then $\mathrm{Br}_{\mathrm{nr},k}(k(G)) = \{0\}$.

In this article, we concentrate on field extensions L over $\mathbb{C}$. Thus we will write $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G))$ for $\mathrm{Br}_{\mathrm{nr},\mathbb{C}}(\mathbb{C}(G))$, because there is no ambiguity of the ground field $\mathbb{C}$. As mentioned before, if $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) \neq \{0\}$, then $\mathbb{C}(G)$ is not stably rational over $\mathbb{C}$.

Theorem 1.1. *(See Saltman [14].) Let p be any prime number. Then there is a group G of order p^9 such that $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) \neq \{0\}$. Consequently $\mathbb{C}(G)$ is not stably $\mathbb{C}$-rational.*

A convenient formula for computing $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G))$ was found by Bogomolov [2, Theorem 3.1]. Using this formula, Bogomolov was able to reduce the group order from p^9 to p^6.

Theorem 1.2. *(See Bogomolov [2, Lemma 5.6].) Let p be any prime number. Then there is a group G of order p^6 such that $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G)) \neq \{0\}$.*

Colliot-Thélène and Ojanguren generalized the notion of the unramified Brauer group to the unramified cohomology group $H^d_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ where $d \geq 2$ [5]; also see Saltman's treatment [15]. Again, if $\mathbb{C}(G)$ is stably $\mathbb{C}$-rational, then $H^d_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) = \{0\}$ [5, Proposition 1.2]. Moreover, $H^2_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G))$.