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On vanishing criteria that control finite group structure

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ABSTRACT

Many results have been established that show how arithmetic conditions on conjugacy class sizes affect group structure. A conjugacy class in G is called vanishing if there exists some irreducible character of G which evaluates to zero on the conjugacy class. The aim of this paper is to show that for some classical results it is enough to consider the same arithmetic conditions on the vanishing conjugacy classes of the group.

1. Introduction

A well-established area of research in finite group theory considers the relationship between the structure of a group G and sets of positive integers which can naturally be associated to G. One of these sets, the set $\mathrm{cs}(G)$ which consists of the conjugacy class sizes of the group G, has led to many structural results about G.

Some of the classical results concerning $\mathrm{cs}(G)$ give arithmetical conditions on $\mathrm{cs}(G)$ which yield that G is either a soluble or supersoluble group (see [4, Theorems 1 and 2]). It is then natural to consider whether the entire data contained in $\mathrm{cs}(G)$ is required to obtain such structural results; in fact one of the classical results in this direction shows

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that if a prime p does not divide the conjugacy class size of any p'-element in G, then G is p-nilpotent (see [2]).

More recently, there has been an interest in studying such structural results based on a refinement of the set $\mathrm{cs}(G)$ by the character table of the group G. Consider $\chi \in \mathrm{Irr}(G)$, an irreducible character of G. A classical result of Burnside says if χ is non-linear, that is $\chi(1) \neq 1$, then there is at least one element g in G such that $\chi(g) = 0$ [13, Theorem 3.15]. In particular, this implies every non-linear row of the character table contains a zero entry. If one considers conjugacy classes, a natural dual to irreducible characters, then g is a central element in G implies that $|\chi(g)| = \chi(1)$ and thus the column corresponding to g cannot contain a zero. However, a non-central column may not contain a zero, for example the 3-cycles in $\mathrm{Sym}(3)$. Therefore we define an element g in G to be a vanishing element if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$. One can now see that a corollary to Burnside's result is that a group has no vanishing elements if and only if the group is abelian.

Unlike with character values, there is not such a clear statement for a column in the character table to contain a zero. For nilpotent groups it was shown by Isaacs, Navarro and Wolf that g is non-vanishing if and only if g lies in the centre of G [14, Theorem B]. For soluble groups they also show if g is non-vanishing then $\bar{g}$ in $G/F(G)$ must be a 2-element [14, Theorem D]. In general, if an element g is non-vanishing in G and the order of g is coprime to 6, then g lies in $F(G)$ [5, Theorem A]. Hence one is led to question whether results based on conjugacy class sizes should still hold if one restricts focus only to those corresponding to vanishing elements.

The aim of this paper is to prove three theorems generalising the classical results mentioned above to vanishing conjugacy classes.

Theorem A. *Let G be a finite group and p a prime dividing the size of G such that if q is any prime dividing the size of G, then q does not divide $p-1$. Suppose that no vanishing conjugacy class size of G is divisible by p^2. Then G is a soluble group.*

Theorem B. *Let G be a finite group and suppose that every vanishing conjugacy class size of G is square free. Then G is a supersoluble group.*

Theorem C. *Let G be a finite group and suppose a prime p does not divide the size of any vanishing conjugacy class size $|x^G|$ for x a p'-element of G. Then G has a normal p-complement.*

In particular, Theorem A has the following corollary.

Corollary 1.1. *Let G be a finite group and suppose that no vanishing conjugacy class size of G is divisible by 4. Then G is a soluble group.*

Theorems A and B form partial vanishing analogues of [4, Theorem 1] and [4, Theorem 2] respectively. Note that the original statements provide more structural in-