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Exact model structures and recollements

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ABSTRACT

We show how to obtain recollements of triangulated categories using the theory of exact model structures from [13]. After noting how the theory relates to well-known notions in the simplest case of Frobenius categories, we apply these ideas to recollements of homotopy categories of chain complexes. In short, we give model categorical explanations for the classical Verdier localization recollement as well as several recollement situations due to Neeman and Murfet.

1. Introduction

This work is a continuation of [14], where the author showed that there is a strong link between recollements of triangulated categories and the cotorsion pairs which serve as abelian model structures for those categories. But the abelian setting is more restrictive than necessary, and so the idea of this paper is that we may relax the hypotheses in the theory from abelian to the more general setting of exact categories, in the sense of Quillen [22]. We offer an alternate proof and statement of the main theorem from [14]

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yielding a recollement from three cotorsion pairs and we give applications showing how some of the most common and interesting recollements in algebraic geometry can be obtained using exact model structures.

As a motivating example, let's consider a well-known recollement. Let R be a ring and denote by $K(R)$ its homotopy category. So $K(R)$ is the category of chain complexes of R-modules with morphisms the homotopy classes of chain maps. Recall that the derived category $\mathcal{D}(R)$ is, by definition, the category obtained from $K(R)$ by formally inverting the homology isomorphisms. This is often written as $\mathcal{D}(R) = K(R)[\mathcal{W}^{-1}]$ where $\mathcal{W}$ is the class of all homology isomorphisms. $K(R)$ and $\mathcal{D}(R)$ each are triangulated categories and the kernel of the quotient functor $K(R) \xrightarrow{Q} \mathcal{D}(R)$ is precisely the full subcategory $\mathcal{E}/\sim$ of all exact complexes (modulo the chain homotopy relation $\sim$). The functor Q has both a left adjoint and a right adjoint. The right adjoint takes a complex to its K-injective resolution while its left adjoint takes a complex to its K-projective resolution. All told, the localization $\mathcal{E}/\sim \ \to K(R) \xrightarrow{Q} \mathcal{D}(R)$ forms the center arrows in a recollement diagram (see Definition 2.2)

$$\mathcal{E}/\sim \quad \begin{smallmatrix}\longleftarrow \\ \longrightarrow \\ \longleftarrow\end{smallmatrix} \quad K(R) \quad \begin{smallmatrix}\longleftarrow \\ \longrightarrow \\ \longleftarrow\end{smallmatrix} \quad \mathcal{D}(R)\ .$$

Although this diagram holds a lot of information, Corollary 6.7 gives a model categorical description for *all* of this information at once. It can be explained quite easily as follows in terms of cotorsion pairs in the category of chain complexes together with the degreewise split short exact sequences: Let $\mathcal{W}$ denote the class of all contractible complexes, let $\mathcal{E}$ denote the class of all exact complexes, $K\mathcal{I}$ the class of all K-injective complexes, $K\mathcal{P}$ the class of all K-projective complexes, and $\mathcal{A}$ the class of all complexes. We point out that the K-injective complexes are those chain complexes X such that any chain map $E \to X$, with E exact, is null homotopic. With respect to the degreewise split short exact sequences, we have a complete cotorsion pair $(\mathcal{E}, K\mathcal{I})$. It is the analog of the more well-known DG-injective cotorsion pair (Exact complexes, DG-injectives), which is a complete cotorsion pair in the category of chain complexes but with respect to the *usual* short exact sequences of chain complexes. The DG-injective complexes are precisely the K-injective complexes X with each component X_n injective. Similar statements hold for the K-projective and DG-projective complexes. Now, again with respect to the degreewise split short exact sequences, we have the following three complete cotorsion pairs. Moreover, each is equivalent to a Quillen model structure for a homotopy category as described.

- $(\mathcal{W}_1, \mathcal{F}_1) = (\mathcal{W}, \mathcal{A})$ = A Quillen model structure for $K(R)$.
- $(\mathcal{W}_2, \mathcal{F}_2) = (K\mathcal{P}, \mathcal{E})$ = A Quillen model structure for $\mathcal{E}/\sim$.
- $(\mathcal{W}_3, \mathcal{F}_3) = (\mathcal{E}, K\mathcal{I})$ = A Quillen model structure for $\mathcal{D}(R)$.

Having these three cotorsion pairs tells us at once that $K(R)$, $\mathcal{E}/\sim$, and $\mathcal{D}(R)$ are triangulated categories. Moreover, the recollement diagram is automatic from the fact that