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Fixed points of endomorphisms on two-dimensional complex tori

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ABSTRACT

In this paper we investigate fixed-point numbers of endomorphisms on complex tori. Specifically, motivated by the asymptotic perspective that has turned out in recent years to be so fruitful in Algebraic Geometry, we study how the number of fixed points behaves when the endomorphism is iterated. Our first result shows that the fixed-points function of an endomorphism on a two-dimensional complex torus can have only three different kinds of behaviours, and we characterize these behaviours in terms of the analytic eigenvalues. Our second result focuses on simple abelian surfaces and provides criteria for the fixed-points behaviour in terms of the possible types of endomorphism algebras.

Introduction

Given a holomorphic map $f : X \to X$ on a complex variety X, one of the natural questions about f is how many fixed points it has. This number may, as expected, vary a

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lot between different endomorphisms, but it is a recurring theme in Algebraic Geometry that one hopes for much more regularity when adopting an asymptotic perspective. Examples for the fruitfulness of this approach are questions about base loci [8], growth of higher cohomology [5], syzygies [7] and Betti numbers [6]. Concerning the question of fixed points, a natural asymptotic point of view consists in considering large iterates f^n of a given map. Specifically, denoting by $\#\,\mathrm{Fix}(f)$ the number of fixed points of a map f, the question becomes:

> *What is the asymptotic behaviour of the fixed-points function*
>
> $$n \mapsto \#\,\mathrm{Fix}(f^n)$$
>
> *where $f^n = f \circ \cdots \circ f$ denotes n-th iterate of f.*

The growth of the fixed-points function is also of interest in purely analytic contexts (e.g. [13]). In the present paper we consider it when f is a holomorphic map on a complex torus. As is customary (cf. [2]) we set $\#\,\mathrm{Fix}(f) = 0$, if the fixed-points set is infinite, i.e., if f fixes an analytic subvariety of positive dimension.

Consider for instance the multiplication map $m_X : X \to X$, $x \mapsto mx$, on a complex torus X of dimension g, for a given integer $m \geqslant 2$. Its fixed points are the $(m-1)$-torsion points, and hence the fixed-points number

$$\#\,\mathrm{Fix}((m_X)^n) = (m^n - 1)^{2g}$$

grows exponentially with n. It is natural to wonder whether this is typical for endomorphisms on complex tori, and what other behaviour, if any, might occur. For two-dimensional complex tori we provide a complete answer:

Theorem 1. *Let X be a two-dimensional complex torus and let $f : X \to X$ be a non-zero endomorphism. Then the fixed-points function $n \mapsto \#\,\mathrm{Fix}(f^n)$ has one of the following three behaviours:*

(B1) *It grows exponentially in n, i.e., there are real constants $A, B > 1$ and an integer N such that for all $n \geqslant N$,*

$$A^n \leqslant \#\,\mathrm{Fix}(f^n) \leqslant B^n.$$

In this case both eigenvalues of f (i.e., of its analytic representation $\rho_a(f) \in M_2(\mathbb{C})$) are of absolute value $\neq 1$.

(B2) *It is a periodic function. In this case the non-zero eigenvalues of f are roots of unity, and they are contained in the set of k-th roots of unity where $k \in \{1, \ldots, 6, 8, 10, 12\}$.*