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Blowing up finitely supported complete ideals in a regular local ring

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ABSTRACT

Let I be a finitely supported complete $\mathfrak{m}$-primary ideal of a regular local ring $(R, \mathfrak{m})$. We consider singularities of the projective models $\operatorname{Proj} R[It]$ and $\operatorname{Proj} \overline{R[It]}$ over $\operatorname{Spec} R$, where $\overline{R[It]}$ denotes the integral closure of the Rees algebra $R[It]$. A theorem of Lipman implies that the ideal I has a unique factorization as a $*$-product of special $*$-simple complete ideals with possibly negative exponents for some of the factors. If $\operatorname{Proj} \overline{R[It]}$ is regular, we prove that $\operatorname{Proj} \overline{R[It]}$ is the regular model obtained by blowing up the finite set of base points of I. Extending work of Lipman and Huneke–Sally in dimension 2, we prove that every local ring S on $\operatorname{Proj} \overline{R[It]}$ that is a unique factorization domain is regular. Moreover, if $\dim S \geq 2$ and S dominates R, then S is an infinitely near point to R, that is, S is obtained from R by a finite sequence of local quadratic transforms.

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1. Introduction

Let $(R, \mathfrak{m})$ be a regular local ring of dimension at least 2. A regular local ring S that dominates R is *infinitely near to* R if $\dim S \geq 2$ and S may be obtained from R by a finite sequence (possibly empty) of local quadratic transforms. An infinitely near point S to R is a *base point* of an ideal I of R if the transform I^S of I in S is a proper ideal of S. The set of base points of an ideal I of R is denoted $\mathcal{BP}(I)$, and the ideal I is said to be *finitely supported* if the set $\mathcal{BP}(I)$ is finite.

The infinitely near points to R form a partially ordered set with respect to domination. The regular local ring R is the unique minimal point with respect to this partial order. For an ideal I of R, the set $\mathcal{BP}(I)$ of base points of I is a partially ordered subset of the set of infinitely near points to R. If the set $\mathcal{BP}(I)$ is finite, we refer to the maximal regular local rings in $\mathcal{BP}(I)$ as *terminal base points* of I. If I is a finitely supported ideal, then results of Lipman [8, Prop. 1.21, Cor. 1.22] imply that $\dim S = \dim R$ and S/I^S is Artinian for each base point S of I. In particular, the ideal I is $\mathfrak{m}$-primary.

Definition 1.1. Let $(R, \mathfrak{m})$ be a regular local ring and let I be a finitely supported $\mathfrak{m}$-primary ideal. Let $\Gamma := \mathcal{BP}(I)$ denote the finite set of base points of I. By successively blowing up the maximal ideals of the points in Γ we obtain a regular projective model[1] X_Γ over R and a projective morphism $X_\Gamma \to \operatorname{Spec} R$. We call X_Γ the *saturated regular model* associated to the ideal I, or more precisely, to the set $\Gamma = \mathcal{BP}(I)$.

The model X_Γ may be obtained by first blowing up the maximal ideal $\mathfrak{m}$ of R to obtain the regular model $\operatorname{Proj} R[\mathfrak{m}t] = X_1$. Each infinitely near point S in $\Gamma = \mathcal{BP}(I)$, other than R, dominates a unique point on the model X_1. The points in Γ in the first neighborhood of R are obtained from R by one local quadratic transform and are points on the model X_1. Each infinitely near point S in $\Gamma \setminus \{R\}$ is either a point on the model X_1 or is an infinitely near point to a unique point S_1, where S_1 is a point on the model X_1. Associated to each infinitely near point $S_1 \in X_1$ such that $\dim S_1 = \dim R$, there exists a unique coherent $\mathcal{O}_{X_1}$-ideal sheaf $\mathcal{I}$ such that the stalk $\mathcal{I}_{S_1}$ is the maximal ideal of $\mathcal{O}_{X_1,S_1}$ and the stalk $\mathcal{I}_T = \mathcal{O}_{X_1,T}$ for each point T in $X_1 \setminus \{S_1\}$ [8, Lemma 2.3].

On X_1, we blow up the ideal sheaf that is the product of the ideal sheaves that correspond to the points $S_1 \in \Gamma \cap X_1$ to obtain the regular model X_2. There exist

[1] We are using the language of Section 17, Chapter VI of Zariski–Samuel [14]. Thus, for R a subring of a field K and A a finitely generated R-subalgebra of K, the *affine model* over R associated to A is the set of local rings A_P, where P varies over the set of prime ideals of A. A *model* M over R is a subset of the local subrings of K that contain R that has the properties: (i) M is a finite union of affine models over R, and (ii) each valuation ring of K that contains R dominates at most one of the local rings in M. This second condition is called *irredundance*. A model M over R is said to be *complete* if each valuation ring of K that contains R dominates a local ring in M. The model M is said to be *projective* over R if there exists a finite set $a_0, a_1, \dots, a_n$ of nonzero elements of K such that M is the union of the affine models defined by the rings $A_i = R[\frac{a_0}{a_i}, \frac{a_1}{a_i}, \dots, \frac{a_n}{a_i}]$, $i = 0, 1, \dots, n$. The models we consider are either affine or projective models over a Noetherian integral domain R. In the language of schemes, these models correspond, respectively, to affine or projective schemes over $\operatorname{Spec} R$.