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The moduli space of complex 5-dimensional Lie algebras ☆

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A B S T R A C T

In this paper, we study the moduli space of all complex 5-dimensional Lie algebras, realizing it as a stratification by orbifolds, which are connected only by jump deformations. The orbifolds are given by the action of finite groups on very simple complex manifolds. Our method of determining the stratification is based on the construction of versal deformations of the Lie algebras, which allow us to identify natural neighborhoods of the elements in the moduli space.

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1. Introduction

Moduli spaces of low dimensional Lie algebras possess very interesting properties. A moduli space of algebras of a fixed dimension has a natural stratification by orbifolds. In the case of complex Lie algebras, these orbifolds consist of quasi-projective spaces, given by removing a divisor from a certain $\mathbb{CP}^n$, together with an action of a symmetric group. Some of the strata are singletons, while other strata can be labeled by projective coordinates. This point of view is new, and uses deformation theory to give the stratification in a unique form, where elements in a stratum deform in a smooth manner along the stratum, jump to elements in different strata, and also deform smoothly along the families of the points to which they jump.

For algebras of dimension less than or equal to 5, we obtained that each family (non-singleton) of solvable Lie algebras contains one special element, which is nilpotent. In fact, the usual picture is that a family is given by the action of a symmetric group on $\mathbb{P}^n$, and the *generic point* of $\mathbb{P}^n$ corresponds to the nilpotent. We remark that if $\mathbb{P}^n$ is given by projective coordinates $(p_0 : \cdots : p_n)$, then the generic point is $(0 : \cdots : 0)$, which is usually excluded from consideration by algebraic geometers, but cannot be excluded in our consideration, since there is an algebra corresponding to the generic point. (The terminology *generic point* is a bit unfortunate, since the generic point behaves in a very nongeneric manner.) The key point here is that the nilpotent elements fit within a larger picture in a natural manner. We also note that the generic element in one family may be isomorphic to the generic element in another family. This behavior represents the only overlap between families.

A big difference between our analysis of the structure of the moduli space of algebras, and classifications given by previous authors is that our goal is not to simply determine a list of the algebras, but to see how the moduli space is glued together. For that, we needed to compute not only the cohomology of the algebras, but *versal deformations* of the algebras, which give complete information about all of the deformations. We will describe later how these versal deformations are computed.

A classification of two- and three-dimensional Lie algebras was first given in [6]. A classification of real four- and five-dimensional algebras was given in [7,8]. A list of the 5-dimensional real Lie algebras was given in [10] and the same list also appears in the recent article [3]. Real and complex Lie algebras are listed up to dimension 6 in the recent work [11]. The main emphasis of these classification methods was to give a complete list of the algebras, without many duplicates, although sometimes the lists were subdivided in terms of other properties, such as invariants of the algebras.

The description we give of the moduli space gives a more natural decomposition of the moduli space, and divides it into fewer pieces. We first give a summary of the moduli spaces of three- and four-dimensional complex Lie algebras, then we give an analysis of how we construct the moduli space of 5-dimensional algebras, and afterwards, we give details on the deformations of the algebras.