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On the commuting variety of a reductive Lie algebra and other related varieties

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ABSTRACT

The nilpotent cone of a reductive Lie algebra has a desingularization given by the cotangent bundle of the flag variety. Analogously, the nullcone of a Cartesian power of the algebra has a desingularization given by a vector bundle over the flag variety. As for the nullcone, the subvariety of elements whose components are in a same Borel subalgebra, has a desingularization given by a vector bundle over the flag variety. In this note, we study geometrical properties of these varieties. For the study of the commuting variety, the analogous variety to the flag variety is the closure in the Grassmannian of the set of Cartan subalgebras. So some properties of this variety are given. In particular, it is smooth in codimension 1. We introduce the generalized isospectral commuting varieties and give some properties. Furthermore, desingularizations of these varieties are given by fiber bundles over a desingularization of the closure in the Grassmannian of the set of Cartan subalgebras contained in a given Borel subalgebra.

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Contents

1. Introduction

In this note, the base field $\Bbbk$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, ℓ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and G is its adjoint group. As usual, $\mathfrak{b}$ denotes a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$, and B the normalizer of $\mathfrak{b}$ in G.

1.1. Main results

Let $\mathcal{B}^{(k)}$ be the subset of elements $(x_1, \ldots, x_k)$ of $\mathfrak{g}^k$ such that $x_1, \ldots, x_k$ are in a same Borel subalgebra of $\mathfrak{g}$. This subset of $\mathfrak{g}^k$ is closed and contains two interesting subsets: the nullcone of $\mathfrak{g}^k$ denoted by $\mathcal{N}^{(k)}$ and the generalized commuting variety of $\mathfrak{g}$ that is the closure in $\mathfrak{g}^k$ of the subset of elements $(x_1, \ldots, x_k)$ such that $x_1, \ldots, x_k$ are in a same Cartan subalgebra of $\mathfrak{g}$. We denote it by $\mathcal{C}^{(k)}$. According to [16, Ch. 2, §1, Theorem], for $(x_1, \ldots, x_k)$ in $\mathcal{B}^{(k)}$, $(x_1, \ldots, x_k)$ is in $\mathcal{N}^{(k)}$ if and only if $x_1, \ldots, x_k$ are nilpotent. According to a Richardson Theorem [18], $\mathcal{C}^{(2)}$ is the commuting variety of $\mathfrak{g}$.

There is a natural projective morphism $G \times_B \mathfrak{b}^k \longrightarrow \mathcal{B}^{(k)}$. For $k = 1$, this morphism is not birational but for $k \geq 2$, it is birational (see Lemma 2.2 and Lemma 2.4). Furthermore, denoting by $\mathcal{X}$ the subvariety of elements (x, y) of $\mathfrak{g} \times \mathfrak{h}$ such that y is in the closure of the orbit of x under G, the morphism

$$G \times \mathfrak{b} \longrightarrow \mathcal{X} , \qquad (g, x) \longmapsto (g.x, \overline{x})$$

with $\overline{x}$ the projection of x onto $\mathfrak{h}$ defines through the quotient a projective and birational morphism $G \times_B \mathfrak{b} \longrightarrow \mathcal{X}$ and $\mathfrak{g}$ is the categorical quotient of $\mathcal{X}$ under the action of $W(\mathcal{R})$ on the factor $\mathfrak{h}$, with $W(\mathcal{R})$ the Weyl group of $\mathfrak{g}$. For $k \geq 2$, the inverse image of $\mathcal{B}^{(k)}$ by the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$ is not irreducible but the canonical action of $W(\mathcal{R})^k$ on $\mathcal{X}^k$ induces a simply transitive action on the set of its irreducible components. Setting:

$$\mathcal{B}_{\mathrm{x}}^{(k)} := \{((g(x_1), \overline{x_1}), \ldots, (g(x_k), \overline{x_k})) \mid (g, x_1, \ldots, x_k) \in G \times \mathfrak{b}^k\},$$