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Equality of elementary linear and symplectic orbits with respect to an alternating form

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An elementary symplectic group w.r.t. an invertible alternating matrix is defined. It is shown that the group of symplectic transvections of a symplectic module coincides with this elementary symplectic group in the free case. Equality of orbit spaces of a unimodular element under the action of the linear group, symplectic group, and symplectic group w.r.t. an invertible alternating matrix is established.

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1. Introduction

Let R denote a commutative ring with 1 in the sequel. This article and the articles [3,4] are all inspired by the article [9] and the famous lemma of L.N. Vaserstein ([9], Lemma 5.5), that

$$e_1\mathrm{E}_{2n}(R) = e_1\{\mathrm{Sp}_{2n}(R) \cap \mathrm{E}_{2n}(R)\} = e_1\{\mathrm{Sp}_{\varphi}(R) \cap \mathrm{E}_{2n}(R)\},$$

where φ is an invertible alternating matrix of size $2n$, and $\mathrm{Sp}_{\varphi}(R)$ is the isotropy group of φ, i.e. $\mathrm{Sp}_{\varphi}(R) = \{\alpha \in \mathrm{SL}_{2n}(R) \mid \alpha^t\varphi\alpha = \varphi\}$.

This is the last in a series of 3 articles inspired by the above equality of L.N. Vaserstein, regarding the equality of elementary linear and symplectic orbits of a unimodular row. The first article appeared in the Journal of K-theory [3] and we showed that if v is a unimodular row of even length $2n$, $n \geq 2$, over a commutative ring R, then $v\mathrm{E}_{2n}(R, I) = v\mathrm{ESp}_{2n}(R, I)$. The second article appeared in the Journal of Pure and Applied Algebra [4] and we showed that if $(Q, \langle,\rangle_{\psi_n})$ is a symplectic module, and $\mathbb{H}(R)$ is the usual hyperbolic space, with rank $(P) \geq 1$, then for a unimodular element $(a, b, p) \in \mathbb{H}(R) \perp P$ the elementary linear and symplectic transvection orbits coincide.

In this article, we touch upon the second equality of L.N. Vaserstein. Let $(Q, \langle,\rangle_{\varphi})$ be a symplectic module w.r.t. an invertible alternating form φ. L.N. Vaserstein gave examples of symplectic matrices w.r.t. an invertible alternating matrix in ([9], Lemma 5.4). We denoted by $\mathrm{ESp}_{\varphi}(R)$ the subgroup of $\mathrm{Sp}_{\varphi}(R)$ generated by these matrices. This will be the elementary symplectic group w.r.t. an invertible alternating matrix φ, when we are dealing with the case where Q is a free module. In the general case, we will define the elementary symplectic group as follows:

Let $(Q, \langle,\rangle)$ be a symplectic R-module with Q finitely generated projective module of even rank. Recall that $\mathrm{Sp}(Q, \langle,\rangle)$ is the group of isometries. We define $V(Q, \langle,\rangle)$ to be the collection of all

$$\begin{aligned}&\{\alpha(1) : \alpha(X) \in \mathrm{Sp}(Q[X], \langle,\rangle_{\otimes R[X]}), \alpha(0) = id., \text{ and}\\ &\alpha(X)_{\mathfrak{p}} \in \mathrm{ESp}_{\varphi\otimes R_{\mathfrak{p}}[X]}(R_{\mathfrak{p}}[X]), \text{ for all } \mathfrak{p} \in \mathit{Spec}(R)\},\end{aligned}$$

where $\langle,\rangle$ correspond to an alternating matrix φ (w.r.t. some basis) of Pfaffian 1 over the local ring $R_{\mathfrak{p}}$ at the prime ideal $\mathfrak{p}$. For a relative version of this definition one can see Definition 6.2. This group can be considered as the generalisation of $\mathrm{ESp}_{\varphi}(R)$ in the case of projective modules. This realisation follows from Lemma 5.18 and Theorem 6.4.

We have recalled the definition of $\mathrm{Um}_n(R, I)$ and $\mathrm{Um}(Q, IQ)$ in the beginning of the next section. For definitions of $\mathrm{E}_n(R, I)$, $\mathrm{ESp}_{2n}(R, I)$, $\mathrm{ESp}_{\varphi}(R, I)$, $\mathrm{ETrans}(Q, IQ)$, and $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle)$ one can see Definitions 2.2, 2.7, 3.4, 5.4, and 5.13 respectively. Our main results regarding equality of orbit spaces in the case of free modules and in the case of projective modules are as follows: