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On the decomposition matrix of the partition algebra in positive characteristic

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ABSTRACT

We examine the structure of the partition algebra $P_n(\delta)$ over a field k of characteristic $p > 0$. In particular, we describe the decomposition matrix of $P_n(\delta)$ when $n < p$ and $\delta \neq 0$, and when $n = p$ and $\delta = p - 1$.

1. Introduction

The partition algebra was originally defined by Martin in [12] over $\mathbb{C}$ as a generalisation of the Temperley–Lieb algebra for δ-state n-site Potts models in statistical mechanics, and independently by Jones [11]. Although this interpretation requires δ to be integral, it is possible to define the algebra for any δ. It was shown in [18] that the partition algebra $P_n^{\mathbb{F}}(\delta)$ over an arbitrary field $\mathbb{F}$ is a cellular algebra, with cell modules $\Delta_\lambda(n)$ indexed by partitions λ of size at most n. If we suppose $\delta \neq 0$, then in characteristic zero these partitions also label a complete set of non-isomorphic simple modules, given by the heads of the corresponding cell modules. In positive characteristic the simple modules

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are indexed by the subset of p-regular partitions (again under the assumption $\delta \neq 0$). It is natural to then ask how the simple modules arise as composition factors of the cell modules. In the case $\operatorname{char} \mathbb{F} = 0$ this has been entirely resolved by Martin [13] and Doran and Wales [3], however there has previously been little investigation into the positive characteristic case.

Martin provides in [13] a condition on λ, μ and δ for when there is a homomorphism in characteristic zero between cell modules labelled by λ and μ, provided $\delta \neq 0$. This was strengthened in [3] to allow for $\delta = 0$. In [1] this condition was reformulated in terms of the reflection geometry of a Weyl group W under a δ-shifted action. By then considering the action of the corresponding affine Weyl group W^p, a description of the blocks of the partition algebra in positive characteristic was given.

In this paper we continue to investigate the representations of $P_n^{\mathbb{F}}(\delta)$ when $\operatorname{char} \mathbb{F} = p > 2$. We show that by placing certain restrictions on the values of n, δ and p we can in these cases compute the decomposition matrix of $P_n^{\mathbb{F}}(\delta)$.

In Section 2 we set up the notation and definitions that will be used throughout the paper, and review some previous results. In Section 3 we recall some results regarding the representation theory of the symmetric group, and the abacus method of representing partitions. Section 4 introduces the partition algebra and recalls the block structure in characteristic zero and in prime characteristic. In Section 5 we obtain the decomposition matrix of the partition algebra in positive characteristic. We separate this last section into three subsections, each dealing with a particular set of values for n and δ.

When writing this paper, it was brought to the author's attention that the decomposition numbers of the partition algebra $P_n^k(\delta)$ over a field k of characteristic $p > n$ were obtained independently, and by different methods, by A. Shalile [17].

1.1. Notation

Throughout this paper, we fix a prime number $p > 2$ and a p-modular system (K, R, k). That is, R is a discrete valuation ring with maximal ideal $P = (\pi)$, field of fractions $\operatorname{Frac}(R) = K$ of characteristic 0, and residue field $k = R/P$ of characteristic p. We will use $\mathbb{F}$ to denote either K or k.

We also fix a parameter $\delta \in R$ and assume that its image in k is non-zero (so in particular, $\delta \neq 0 \in R$). We will use δ to denote both the element in R and its projection in k.

2. Preliminaries

Suppose A is an R-algebra, free and of finite rank as an R-module. We can extend scalars to produce the K-algebra $A_K = K \otimes_R A$ and the k-algebra $A_k = k \otimes_R A$. Given an A-module M, we can then also consider the A_K-module $M_K = K \otimes_R M$ and the A_k-module $M_k = k \otimes_R M$.