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On totally decomposable algebras with involution in characteristic two

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ABSTRACT

A necessary and sufficient condition for a central simple algebra with involution over a field of characteristic two to be decomposable as a tensor product of quaternion algebras with involution, in terms of its Frobenius subalgebras, is given. It is also proved that a bilinear Pfister form, recently introduced by A. Dolphin, can classify totally decomposable central simple algebras of orthogonal type.

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1. Introduction

An old result due to A.A. Albert states that every central simple algebra A of degree 4 which carries an involution of the first kind can be decomposed as a tensor product of two quaternion algebras (see [18, §16]). This result is no longer valid if A is of degree 8 by the examples given in [1] over fields of characteristic different from 2 and in [26] over fields of characteristic 2. In [1], it was also shown that if A is of degree 2^n over a field of characteristic different from 2, then A decomposes into a tensor product of quaternion algebras if and only if there exists a finite square-central subset of A (called a *q-generating set*) which satisfies some commuting properties. Over a field of particular cohomological dimension, it is known that central simple algebras which carry an involution of the first kind can be decomposed as a tensor product of quaternion algebras (see [15], and [4] for a characteristic 2 counterpart). In [3], a similar result was proved under the hypothesis that the base field has the u-invariant less than or equal to 8.

A closely related problem is to determine the conditions under which a central simple algebra with involution (A, σ) is *totally decomposable* (i.e., (A, σ) decomposes as a tensor product of σ-invariant quaternion algebras). In [25], it was shown that if A is of degree 4 over a field of characteristic different from 2 and σ is of symplectic type, then A can be decomposed as a tensor product of two σ-invariant quaternion algebras. A proof of this result in characteristic 2 was given in [27], also a characteristic independent proof of this result and a criterion for decomposability in the case where σ is orthogonal can be found in [19]. A similar criterion for the unitary case of degree 4 and of arbitrary characteristic was derived in [16]. A cohomological invariant to detect decomposability for degree 8 algebras with symplectic involution over a field of characteristic different from 2 can be found in [12]. For the case of degree 8 algebras with orthogonal involution (A, σ) over a field of characteristic different from 2, a criterion for decomposability in terms of the Clifford algebra of (A, σ) can be found in [18, (42.11)], see also [28, (3.10)]. If A is split and of degree 2^n over a field of characteristic different from 2, by [5] an involution σ of orthogonal type on A is totally decomposable if and only if it is adjoint to a bilinear Pfister form (see [21] and [10] for a characteristic 2 counterpart of this result).

Another relevant problem is to find invariants which classify involutions on a given central simple algebra up to conjugation. Orthogonal involutions of degree $\leqslant 4$ in characteristic different from 2 can be classified by their Clifford algebras [18, §15], [20, §2]. A degree 4 central simple algebra with symplectic involution (A, σ) can be classified by a 3-fold Pfister form or an Albert form associated to σ, see [17] and [18, §16].

In this work we study the problems of decomposition and classification of central simple algebras with involution in the case of characteristic 2. In 4.6, we show that a central simple algebra with involution (A, σ) over a field of characteristic 2 is totally decomposable if and only if there exists a symmetric and self-centralizing subalgebra $S = \Phi(A, \sigma)$ of A such that (i) $x^2 \in F$ for every $x \in S$ and (ii) $\dim_F S = 2^{r_F(S)}$, where $r_F(S)$ is the minimum rank of S. In the case where (A, σ) is totally decomposable central simple algebra with involution of orthogonal type we show that the aforementioned subalgebra